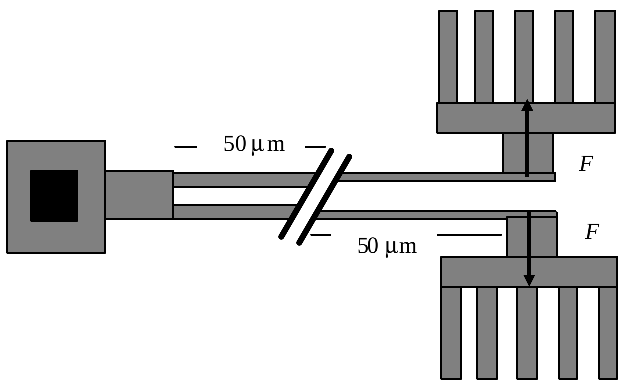
UNIVERSITY OF CALIFORNIA AT BERKELEY
College of Engineering
Dept. of Electrical Engineering and Computer Sciences

Problem Set #4
Due Thursday, October 9, 2003, 5:00 pm

EECS C245 / ME C218 **Fall 2003**

1. The layout below shows a single-ended, comb-drive tuning fork resonator made from a 2 μm-thick polysilicon film. The length of the tines is 100 μm and their thickness is $h = 2$ μm in the bending direction for the first 50 μm from the shoulder of the tuning fork, after which the thickness is 1 μm. The area of the interdigitated comb fingers and their attachment structures at the end of each tine is 168 μm^2.



a) Use the Principle of Virtual Work to estimate the linear spring constant for one of the tines, when deflected by the force F.

a) assume $y(x) = c_2 x^2 + c_3 x^3$

strain energy: $W_{bend} = \int_0^L \frac{1}{2} E I_z(x) \left(\frac{d^2y}{dx^2}\right)^2 dx$

$$= \frac{1}{2} E \frac{h}{12} \int_0^L W^3(x) \cdot (2c_2 + 6c_3 x)^2 dx$$

$$W(x) = \begin{cases} W, & 0 < x < L/2 \\ W/2, & L/2 < x < L \end{cases}$$

$$= \frac{EW^3 h}{24} \int_0^{L/2} (2c_2 + 6c_3 x)^2 dx + \frac{E(W/2)^3 h}{24} \int_{L/2}^{L} (2c_2 + 6c_3 x)^2 dx$$

$$W_{bend} = \left(\frac{EW^3 h}{128}\right) \left\{ 12 c_2^2 L + 22 c_2 c_3 L^2 + 15 c_3^2 L^3 \right\}$$

Potential energy:

$$U = W_{bend} - F \cdot \overbrace{y(L)}^{c_2 L^2 + c_3 L^3} \quad \text{for point load at end of beam}$$

$$\begin{cases} \frac{\partial U}{\partial c_2} = 0 = \frac{EW^3 h}{128} \left\{ 24 c_2 L + 44 c_3 L^2 \right\} - FL^2 \\ \frac{\partial U}{\partial c_3} = 0 = \frac{EW^3 h}{128} \left\{ 44 c_2 L^2 + 30 c_3 L^3 \right\} - FL^3 \end{cases}$$

$$c_2 = \frac{256}{59} \left(\frac{FL}{EW^3 h}\right) \; ; \quad c_3 = \frac{64}{59} \frac{F}{EW^3 h}$$

Spring constant $k = \frac{F}{y(L)} = \frac{F}{\frac{256}{59}\left(\frac{FL^3}{EW^3 h}\right) + \frac{64}{59}\left(\frac{FL^3}{EW^3 h}\right)} = \left(\frac{59}{320}\right) \frac{EW^3 h}{L^3}$

* note softer than uniform-thickness cantilever

$\underline{k = 0.47 \, \mu N/\mu m}$

b) Estimate a tine's effective mass, for a fundamental mode. You can use your deflection function from part (a) as the mode shape.

b) $y(x,t) = y(x) \cdot \cos\omega t$

$$K_{max} = \int_0^L \tfrac{1}{2}\rho\, W(x) h\, \omega^2 \cdot y^2(x)\,dx$$

$$y(x) = \frac{256}{59}\frac{FL}{EW^3h}\cdot x^2 + \frac{64}{59}\frac{F}{EW^3h}x^3$$

$$= \underbrace{\frac{64}{59}\left(\frac{F}{EW^3h}\right)}_{A}\cdot\left[4L\cdot x^2 + x^3\right] = A x^2(x+4L)$$

$$y^2(x) = A^2\left[x^4(x^2+8Lx+16L^2)\right]$$

$$= A^2(x^6+8Lx^5+16L^2x^4)$$

$$K_{max} = \tfrac{1}{2}\rho W h\omega^2 A^2\left\{\int_0^{L/2}(x^6+8Lx^5+16L^2x^4)dx + \tfrac{1}{2}\int_{L/2}^{L}(x^6+8Lx^5+16L^2x^4)dx\right\}$$

$$K_{max} = \tfrac{1}{2}\rho W h\omega^2\left(\frac{64}{59}\right)^2\left(\frac{F}{EW^3h}\right)^2(2.40\,L^7)$$

Define M_{eff} of beam: $K_{max} = \tfrac{1}{2}M_{eff}\,\omega^2 \underbrace{y^2(L)}_{(320/59)^2\left(\frac{FL^3}{EW^3h}\right)^2}$

Solve for $M_{eff} = \frac{1}{25}(2.4)\rho W h L$

Total effective mass $= M_{eff\,TOT} = M_{comb} + M_{eff}$

$$= \rho\cdot \underbrace{A_{comb}}_{168\,\mu m^2}\cdot h + \frac{2.4}{25}\rho W h L$$

$\underline{M_{eff,TOT} = 0.87\,ng}$

c) Estimate the fundamental resonant frequency f_1 of the tuning fork using the Rayleigh-Ritz method.

c) $K_{max} = W_{max}$ from (b) and (a)

$$\omega = \sqrt{k/M_{eff,TOT}} = \left[\frac{0.47\,N/m}{0.87\times10^{-9}\times10^{-3}kg}\right]^{1/2} = 735\,krad\,s^{-1}$$

$\underline{f = \frac{1}{2\pi}\omega = 117\,kHz}$

2. Double-ended tuning forks (DETF) are potentially useful as frequency references for communication circuits. As the amplitude of vibration increases, the tuning fork's deflection is no longer proportional to the applied load. In this problem, use the mode shapes provided in Lecture 9 as trial functions. You can neglect the residual stress in the tines.

   a) Given that the DETF is driven in the fundamental anti-symmetric mode and that the displacement at the center of the beam is y, use the principle of virtual work to estimate the dependence of the applied force F (concentrated in the beam's center) on y for large displacements. Your answer should include a term proportional to y^3.

② a) Use normalized mode shape from lecture:

$$y_1(x) = c\kappa\left[\sinh\beta\epsilon - \sin\beta\epsilon + \alpha(\cosh\beta\epsilon - \cos\beta\epsilon)\right]; \quad \epsilon = x/L$$

↑ parameter for variation of U.

$$\int_0^L \left(\frac{d^2y_1}{dx^2}\right)^2 dx = \frac{1}{L^3}\int_0^1 \left(\frac{d^2y_1(\epsilon)}{d\epsilon^2}\right)^2 d\epsilon = \frac{198.5c^2}{L^3}$$

$$\int_0^L \left(\frac{dy_1}{dx}\right)^2 dx = \frac{1}{L}\int_0^1 \left(\frac{dy_1(\epsilon)}{d\epsilon}\right)^2 d\epsilon = \frac{4.85c^2}{L}$$

Strain in beam (following lecture): $\epsilon_x = \epsilon_{bend} + \underbrace{\epsilon_{x,axial}}_{\text{self-generated, no S.}}$

$$= -y'\frac{d^2y}{dx^2} + \frac{1}{2L}\int_0^L \left(\frac{dy}{dx}\right)^2 dx$$

$$W = \frac{1}{2}EW\int_0^L\int_{-h/2}^{h/2} \epsilon_x^2\, dy'\, dx$$

$$= \frac{1}{2}EW\int_0^L \left[\frac{h^3}{12}\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{4.85}{2L^2}\right)^2 c^4 h\right] dx$$

$$= \frac{1}{2}EW\left[\frac{h^3}{12L^3}(198.5c^2) + 5.88\,\frac{h}{L^3}c^4\right]$$

Require $\frac{\partial U}{\partial c} = \frac{\partial}{\partial c}(W - F\cdot c) = 0$

$$\therefore\ F = 16.54\left(\frac{EWh^3}{L^3}\right)\underbrace{c}_{y_1} + 23.52\left(\frac{EWh}{L^3}\right)\underbrace{c^3}_{y_1^3}\ ; \quad y_1(L/2) = c\phi(1/2) = c$$

b) Now consider that the DETF is driven in its *second mode* by concentrated forces (each called *F*) applied at the locations of the peak displacement *y*. What is the functional relationship between *F* and *y*?

b) Corrected coefficients for second mode: $\alpha = -0.99$, $\beta = 7.85$, $\kappa = -0.663$

$$\int_0^1 \left(\frac{d^2y_2(\epsilon)}{d\epsilon^2}\right)^2 d\epsilon = 1676\,c^2 \qquad \int_0^1 \left(\frac{dy_2(\epsilon)}{d\epsilon}\right)^2 d\epsilon = 20.1\,c^2$$

∴ analysis is the same as for (a) with new integrals.

$$W = \frac{1}{2}EW\left[\frac{h^3}{12L^3}\cdot 1676c^2 + \frac{(20.1)^2}{4L^3}hc^4\right]$$

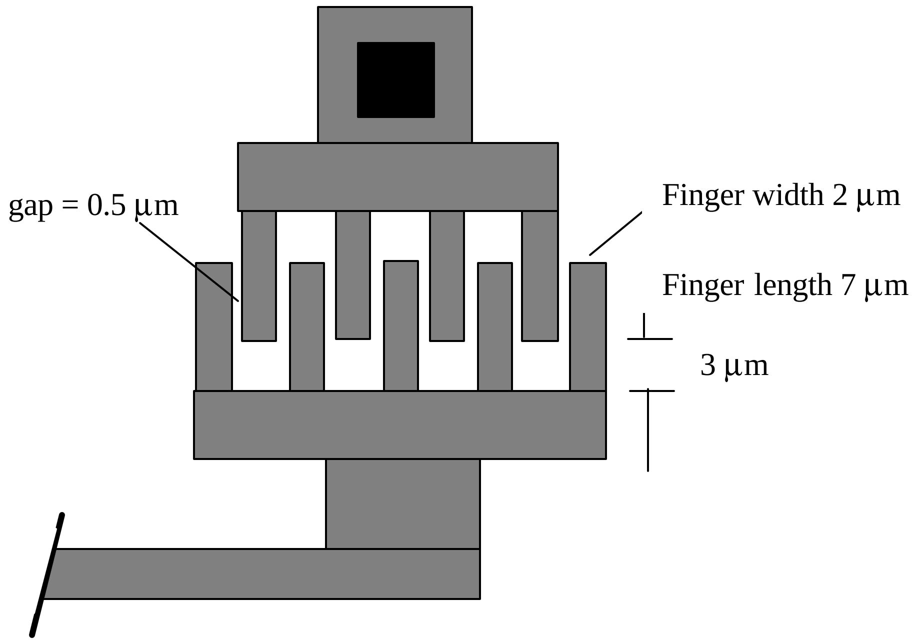
$$\frac{\partial U}{\partial c} = \frac{\partial[W - 2Fc^{*}]}{\partial c} = 0 \qquad * \text{ note work done is } 2Fc$$

$$\therefore\ \frac{1676}{12}\left(\frac{EWh^3}{L^3}\right)c + (20.1)^2\frac{EWh}{L^3}c^3 = 2F.$$

$$F = 69.8\left(\frac{EWh^3}{L^3}\right)y_2 + 202\left(\frac{EWh}{L^3}\right)y_2^3$$

↑ factor of ~4x higher than for y_1 ↑ factor of ~8x higher than for y_1

3. The layout below shows a detailed view of the complete comb-drive from Prob. 1. The tuning fork is suspended 2 μm above the substrate.



a) The tuning fork voltage is V_P = 5 V and a drive voltage $v_d(t)$ = 5 mV $\cos(2\pi f_1 t)$ is applied to the upper tine's comb drive. Find the drive force $f_d(t)$ at the frequency f_1.

3. a) $V_P = 5\,V$, $v_d(t) = 5\,mV \cos\omega t$

$$F_e = \frac{1}{2} V_{rs}^2 \frac{dC_{rs}}{dx}, \quad C_{rs} = \frac{2N\epsilon_0 t}{g}\cdot x; \quad N = 4,\ t = 2\,\mu m,\ g = 500\,nm.$$

$$= \frac{1}{2}\left(V_P - v_d(t)\right)^2 \left(\frac{2N\epsilon_0 t}{g}\right)$$

$$= \frac{N\epsilon_0 t}{g} V_P^2 - \frac{2N\epsilon_0 t}{g} V_P \hat{v}_d \cos\omega t + \left(\frac{N\epsilon_0 t}{g}\right)\hat{v}_d^2 \underbrace{\cos^2\omega t}.$$

neglect small contribution to DC force $\rightarrow \left\{ \frac{1+\cos 2\omega t}{2} \right.$

$$F_e(t) = \underbrace{3.54\,nN}_{DC} - \underbrace{7.08\,pN}_{\text{drive force amplitude}} \cos\omega t.$$

b) Using the Couette and squeeze-film models from Lecture 9, estimate the quality factor of this resonator at a pressure of 100 mTorr.

b) Note: W. Clark's Ph.D. thesis has a typo for μ_p: the correct value is $3.7\times10^{-2}\ kg\,m^{-2}s^{-1}Torr^{-1}$

$$b_{\substack{Couette\\substrate}} = \frac{\mu A}{y_o} = \frac{\mu_p\, p\, \cancel{y_o}\, A}{\cancel{y_o}} \qquad A = 168\mu m^2,\ y_o = 2\mu m.$$

$$b_{\substack{Couette\\fingers}} = \mu_p\, p\, A_f \qquad A_f = (4\times 2\mu m^2)\times 8 = 64\mu m^2$$

$$b_{Couette} = \mu_p\, p\,(A + A_f) = (\underbrace{3.7\times10^{-2}}_{kg\,m^{-2}s^{-1}Torr^{-1}})(\underbrace{0.1}_{Torr})(168+64)\times10^{-12}\,m^2$$

$$= 8.58\times10^{-13}\ kg\,s^{-1}$$

$$b_{squeeze} = \frac{7\mu_p \overbrace{z_o}^{2\mu m}{}^2 A p}{\underbrace{y_o}_{3\mu m}{}^2} = \frac{7\,(3.7\times10^{-2})(2\times10^{-6})^2(7\times4\times10^{-12})(0.1)}{(3\times10^{-6})^2}$$

$$= 0.322\times10^{-12}\ kg\,s^{-1} \quad (<50\%\text{ of Couette})$$

$$Q = \frac{M_{eff}\,\omega_1}{b_{TOT}} = \frac{(0.87\times10^{-12}kg)(0.735\times10^{6}\,rad\,s^{-1})}{(0.858+0.322)\times10^{-12}\,kg\,s^{-1}}$$

$$\underline{Q = 5.4\times10^{5}} \text{ - estimate is high.}$$

c) Find the DC displacement of the tine and the amplitude of its resonance from your answers in (a) and (b).

DC displacement : $y = F_{DC}/k = \dfrac{3.54nN}{0.47nN/nm} = 7.5nm.$

$$\hat{y} = Q\left(F_{AC}/k\right) = (5.4\times10^5)\underbrace{\left(\frac{7.08pN}{0.47nN/nm}\right)}_{15pm}$$

$$\underline{\hat{y} = 810\,nm = 0.81\mu m.}$$

d) Assuming that the lower tine has the same amplitude as the upper tine (but opposite phase), find the current through the comb electrode at resonance.

$$i_s = V_P\frac{dC_s}{dt} = V_P\underbrace{\frac{dC_s}{dy}}\frac{dy}{dt} \qquad y(t) = \hat{y}\cos\underbrace{\omega}_{\omega=\omega_1} t$$

$$\frac{N\epsilon_o h}{g_o} = 283\,pF/m$$

$$i_s(t) = \underbrace{-\omega}_{735\times10^5 s^{-1}}\, 5V\cdot\left(283\tfrac{pF}{m}\right)\cdot(810\times10^{-9}m)\ \sin\omega t$$

$$\underline{i_s(t) = -(0.84nA)\sin\omega t}$$