Lecture Outline

- Reading

- Today’s Lecture
  - Basic Fluidic Concepts
  - Conservation of Mass $\rightarrow$ Continuity Equation
  - Newton’s Second Law $\rightarrow$ Navier-Stokes Equation
  - Incompressible Laminar Flow in Two Cases
  - Squeeze-Film Damping in MEMS
### Viscosity

- Fluids deform continuously in presence of shear forces
- For a Newtonian fluid,
  - Shear stress = Viscosity $\times$ Shear strain
  - $N/m^2 \equiv (N/s/m^2) \times (m/s)/(1/m)$
  - Centipoise = dyne s/cm$^2$
  - No-slip at boundaries

\[ \eta_{\text{air}} = 1.8 \times 10^{-5} N \text{s/m}^2 \]
\[ \eta_{\text{water}} = 8.91 \times 10^{-4} \]
\[ \eta_{\text{lager}} = 1.45 \times 10^{-3} \]
\[ \eta_{\text{honey}} = 11.5 \]

### Density

- Density of fluid depends on pressure and temperature…
  - For water, bulk modulus =
  - Thermal coefficient of expansion =
  - …but we can treat liquids as incompressible

- Gases are compressible, as in Ideal Gas Law

\[ PV = nRT \]
\[ P = \rho \left( \frac{R}{M_w} \right) T \]
Surface Tension

- Droplet on a surface
- Capillary wetting

\[ b = \frac{2\gamma \cos \theta}{\rho gr} \]

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Conservation of Mass

- Control volume is region fixed in space through which fluid moves
  
  \[ \text{rate of accumulation} = \text{rate of inflow} - \text{rate of outflow} \]

- Rate of accumulation

- Rate of mass efflux

\[
\int \int \int \frac{\partial \rho_m}{\partial t} dV + \int \int \rho_m \mathbf{U} \cdot \mathbf{n} dS = 0
\]
Operators

- Gradient and divergence

\[ \nabla \mathbf{U} = \frac{\partial U}{\partial x} \mathbf{i} + \frac{\partial U}{\partial y} \mathbf{j} + \frac{\partial U}{\partial z} \mathbf{k} \]

\[ \nabla = \frac{\partial}{\partial x} \mathbf{e}_x + \frac{\partial}{\partial y} \mathbf{e}_y + \frac{\partial}{\partial z} \mathbf{e}_z \]

\[ \nabla \cdot \mathbf{U} = \text{div} \ \mathbf{U} = \frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} \]

Continuity Equation

- Convert surface integral to volume integral using Divergence Theorem

- For differential control volume

\[ \iiint \left( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) \right) dV = 0 \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0 \]

Continuity Equation
Continuity Equation

- Material derivative measures time rate of change of a property for observer moving with fluid

\[
\frac{\partial \rho}{\partial t} + (\mathbf{U} \cdot \nabla) \rho + \rho \nabla \cdot \mathbf{U} = 0 \\
\frac{D \rho}{D t} = \frac{\partial \rho}{\partial t} + (\mathbf{U} \cdot \nabla) \rho
\]

\[
\frac{D}{D t} = \frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla = \frac{\partial}{\partial t} + U_x \frac{\partial}{\partial x} + U_y \frac{\partial}{\partial y} + U_z \frac{\partial}{\partial z}
\]

For incompressible fluid

\[
\frac{D \rho}{D t} + \rho \nabla \cdot \mathbf{U} = 0 \\
\frac{D \rho}{D t} + \rho \nabla \cdot \mathbf{U} = 0
\]

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Newton’s Second Law for Fluidics

- Newton’s 2nd Law:
  - Time rate of change of momentum of a system equal to net force acting on system

\[
\sum F = \frac{dP}{dt} = \frac{d}{dt} \iiint_S \mathbf{U} \rho dV + \iiint_S \mathbf{U} \rho (\mathbf{U} \cdot \mathbf{n}) dS
\]

Net momentum accumulation rate  
Net momentum efflux rate

Momentum Conservation

- Sum of forces acting on fluid

\[
\sum F = \iiint (-P\mathbf{n} + \tau) dS + \iiint \rho g dV = \frac{d}{dt} \iiint \rho \mathbf{U} dV + \iiint \rho \mathbf{U} \mathbf{U} \cdot \mathbf{n} dS
\]
Navier-Stokes Equation

- Convert surface integrals to volume integrals

\[
\int_s \tau dS = \iiint_V \left( \eta \nabla^2 \mathbf{U} + \frac{\eta}{3} \nabla (\nabla \cdot \mathbf{U}) \right) dV \\
\int_s -P \mathbf{n} dS = \iiint_V -\nabla P dV \\
\int_s \rho \mathbf{U} (\mathbf{U} \cdot \mathbf{n}) dS = \iiint_V \rho \mathbf{U} (\nabla \cdot \mathbf{U}) dV
\]

Navier-Stokes Differential Form

\[
\iiint_V \left(-\nabla P + \rho \mathbf{g} + \eta \nabla^2 \mathbf{U} + \frac{\eta}{3} \nabla (\nabla \cdot \mathbf{U}) \right) dV = \\
\iiint_V \left( \rho \frac{\partial \mathbf{U}}{\partial t} \right) dV + \iiint_V \rho \mathbf{U} (\nabla \cdot \mathbf{U}) dV \\
\iiint_V \left( \rho \frac{D\mathbf{U}}{Dt} \right) dV
\]

\[
\rho \frac{D\mathbf{U}}{Dt} = -\nabla P + \rho \mathbf{g} + \eta \nabla^2 \mathbf{U} + \frac{\eta}{3} \nabla (\nabla \cdot \mathbf{U})
\]
Incompressible Laminar Flow

- Incompressible fluid $\nabla \cdot \mathbf{U} = 0$

\[
\rho \frac{D\mathbf{U}}{Dt} = -\nabla P + \rho \mathbf{g} + \eta \nabla^2 \mathbf{U} + \frac{\eta}{3} \nabla (\nabla \cdot \mathbf{U})
\]

\[
\rho \frac{D\mathbf{U}}{Dt} = -\nabla P + \rho \mathbf{g} + \eta \nabla^2 \mathbf{U}
\]

Cartesian Coordinates

\[
\rho \frac{D\mathbf{U}}{Dt} = -\nabla P + \rho \mathbf{g} + \eta \nabla^2 \mathbf{U}
\]

x direction

\[
\rho \left( \frac{\partial U_x}{\partial t} + U_x \frac{\partial U_x}{\partial x} + U_y \frac{\partial U_x}{\partial y} + U_z \frac{\partial U_x}{\partial z} \right) = -\frac{\partial P}{\partial x} + \rho g_x + \eta \left( \frac{\partial^2 U_x}{\partial x^2} + \frac{\partial^2 U_x}{\partial y^2} + \frac{\partial^2 U_x}{\partial z^2} \right)
\]
Dimensional Analysis

\[
\frac{DU}{Dt} = -\frac{\nabla P}{\rho} + g + \frac{\eta \nabla^2 U}{\rho}
\]

- Each term has dimension \(L/t^2\) so ratio of any two gives dimensionless group
  - inertia/viscous \(= \) Reynolds number
  - pressure/inertia \(= \) \(P/\rho U^2\) = Euler number
  - flow vsound \(v \) \(= \) Mach number

- In geometrically similar systems, if dimensionless numbers are equal, systems are dynamically similar

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Two Laminar Flow Cases

- High \(P\) vs Low \(P\)
- \(U\), \(U_x\), \(U_{\text{max}}\), \(h\), \(\tau_w\), \(\tau_w\)
**Couette Flow**

- Couette flow is steady viscous flow between parallel plates, where top plate is moving parallel to bottom plate.
- No-slip boundary conditions at plates

\[
\frac{DU}{Dt} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + g_x + \frac{\eta}{\rho} \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right)
\]

\[
\frac{\partial^2 U}{\partial y^2} = 0, \quad \text{and} \quad U_x = 0 \text{ at } y = 0
\]

\[
U_x = U \text{ at } y = b
\]

**Shear stress acting on plate due to motion, \( \tau \), is dissipative**

**Couette flow is analogous to resistor with power dissipation corresponding to Joule heating**

\[
\tau_w = -\eta \frac{\partial U}{\partial y} \bigg|_{y=b} = \frac{\tau_w A}{U} = \frac{\eta A}{b}
\]

\[
P_{\text{Couette}} = RU^2
\]
Poiseuille Flow

- Poiseuille flow is a pressure-driven flow between stationary parallel plates.
- No-slip boundary conditions at plates.

\[
\frac{DU_x}{Dt} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + g_x + \eta \left( \frac{\partial^2 U_x}{\partial x^2} + \frac{\partial^2 U_x}{\partial y^2} \right)
\]

\[
\frac{\partial P}{\partial x} = -\frac{\Delta P}{L}, \quad \frac{\partial^2 U_x}{\partial y^2} = -\frac{\Delta P}{L\eta}, \quad U_x = 0 \text{ at } y = 0, b
\]

\[
U_x = \frac{\Delta P}{2L}\eta
\]

Poiseuille Flow

- Volumetric flow rate \( Q \sim [h^3] \)
- Shear stress on plates, \( \tau \), is dissipative.
- Force balance
  - Net force on fluid and plates is zero since they are not accelerating.
  - Fluid pressure force \( \Delta PWh \) (+x) balanced by shear force at the walls of \( 2\tau WL \) (-x).
  - Wall exerts shear force (-x), so fluid must exert equal and opposite force on walls, provided by external force.

\[
Q = W \int_0^b U_x \, dy = \frac{\Delta Phb}{2L}
\]

\[
\tau_w = -\eta \frac{\partial U_x}{\partial y} \bigg|_{y=b} = \frac{\Delta Phb}{2L}
\]

\[
\vec{U} = \frac{Q}{Wb} = \frac{\Delta Phb}{12\eta L} = \frac{1}{4} U_{max}
\]
Poiseuille Flow

- Flow in channels of circular cross section
  \[ U_x = \left( \frac{r_o^2 - r_i^2}{4} \right) \frac{\Delta P}{4 \eta L} \quad Q = \frac{\pi r_o^4 \Delta P}{32 \eta L} \]

- Flow in channels of arbitrary cross section
  \[ D_b = \frac{4 \times \text{Area}}{\text{Perimeter}} \]
  \[ \Delta P = f_D \left( \frac{1}{2} \rho \sigma^2 \right) \frac{L}{D_b} \quad f_D \text{Re}_D = \text{dimensionless constant} \]

- Lumped element model for Poiseuille flow
  \[ R_{pois} = \frac{\Delta P}{Q} = \frac{12 \eta L}{W b^3} \]

Velocity Profiles

- Velocity profiles for a combination of pressure-driven (Poiseuille) and plate motion (Couette) flow

- Stokes, or creeping, flow
  - If Re \( \ll 1 \), inertial term may be neglected compared to viscous term

- Development length
  - Distance before flow assume steady-state profile
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Squeezed Film Damping

• Squeezed film damping in parallel plates
  • Gap $h$ depends on $x$, $y$, and $t$
  • When upper plate moves downward, $P_{\text{air}}$ increases and air is squeezed out
  • When upper plate moves upward, $P_{\text{air}}$ decreases and air is sucked back in
  • Viscous drag of air during flow opposes mechanical motion
Squeezed Film Damping

- Assumptions
  - Gap \( h << \) width of plates
  - Motion slow enough so gas moves under Stokes flow
  - No \( \Delta P \) in normal direction
  - Lateral flow has Poiseuille like velocity profile
  - Gas obeys Ideal Gas Law
  - No change in \( T \)

  \[
  12\eta \frac{\partial(Pb)}{\partial t} = \nabla \cdot [(1 + 6K_n)b^3\nabla P]
  \]

- Reynold's equation
  - Navier-Stokes + continuity + Ideal Gas Law
  - Knudsen number \( K_n \) is ratio of mean free path of gas molecules \( \lambda \) to gap \( h \)
    - \( K_n < 0.01 \), continuum flow
    - \( K_n > 0.1 \), slip flow becomes important
  - \( 1 \) \( \mu m \) gap with room air, \( \lambda = \)

Squeezed Film Damping

- Approach
  - Small amplitude rigid motion of upper plate, \( h(t) \)
  - Begin with non-linear partial differential equation
  - Linearize about operating point, average gap \( h_0 \) and average pressure \( P_0 \)
  - Boundary conditions
    - At \( t = 0 \), plate suddenly displaced vertically amount \( z_0 \) (velocity impulse)
    - At \( t > 0^+ \), \( v = 0 \)
    - Pressure changes at edges of plate are zero: \( \frac{dP}{dh} = 0 \) at \( y = 0 \), \( y = W \)
General Solution

- Laplace transform of response to general time-dependent source $z(s)$

$$F(s) = \left[ \frac{96\eta LW^3}{\pi^4 b_0^3} \sum \frac{1}{\omega_n^4 \left(1 + \frac{s}{\omega_n}\right)} \right] z'(s)$$

$\frac{b}{1 + \frac{s}{\omega_c}}$ 1st term for small amplitude oscillation

$F(s) = \frac{b}{1 + \frac{s}{\omega_c}} z'(s)$,  $b = \frac{96\eta LW^3}{\pi^4 b_0^3}$,  $\omega_c = \frac{\pi^2 b_0^2 P_0}{12\eta W^2}$

damping constant  cutoff frequency

Squeeze Number

- Squeeze number $\sigma_d$ is a measure of relative importance of viscous forces to spring forces
  - $\omega < \omega_c$: model reduces to linear resistive damping element
  - $\omega > \omega_c$: stiffness of gas increases since it does not have time to squeeze out

$squeeze \ number \ \sigma_d = \frac{\omega \omega_c}{\omega^2} = \frac{12\eta W^2}{b_0^2 P_0}$

$\omega_c$
Examples

- Analytical and numerical results of damping and spring forces vs. squeeze number for square plate
- Transient responses for two squeeze numbers, 20 and 60