An Equivalent Circuit Model for Curved Piezoelectric Micromachined Ultrasonic Transducers with Spherical-shape Diaphragms

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Abstract—An equivalent circuit model is analytically derived for a clamped piezoelectric elastic shell with a spherical curved shape. The lumped parameters of the electromechanical elements are extracted from the system equations, and the analytical circuit model is compared with acoustic-piezoelectric FEM simulation results. Very good consistencies are achieved in the frequency responses, electromechanical coupled variations, and the device input impedances.

Keywords—piezoelectric elastic shells; equivalent circuit model; piezoelectric micromachined ultrasonic transducer, curved pMUT

I. INTRODUCTION

MEMS micromachined ultrasonic transducers (MUTs) have a wide range of potential applications such as: medical imaging, gesture recognition, and nondestructive testing. Both cMUTs (capacitive micromachined ultrasonic transducers) and pMUTs (piezoelectric micromachined ultrasonic transducers) have been developed in recent years. While pMUTs require low DC voltages with good linearity, they typically have low electromechanical coupling factor ($k_{eff}^2$) [1,2]. Curved pMUTs have been proposed and demonstrated to achieve orders of magnitude higher electromechanical coupling factors and responses than the state-of-the-art planar pMUTs [3-5] both analytically [6] and experimentally [7]. This work further investigates the equivalent circuit model for curved pMUTs.

Equivalent circuit of an ultrasonic transducer is very important for the prediction of the resonant frequency, the input impedance, and the output pressure under various acoustic loads for performance prediction, design optimization, and interface with ASIC. Not only such theoretical modeling is faster than numerical simulation, but it also provides explicit expressions for the device parameters and offers guidelines for optimal designs. Several groups have reported modeling efforts of various MUTs. For instance, Lohfink et al. described the derivation of a 1D model for cMUT arrays from FEM simulations by the piston radiator and plate capacitance theory [8]. Prasad et al. presented an analytical two-port, lumped-element model of a piezoelectric composite circular plate using classical laminated plate theory (CLPT) [9]. Sammoura et al. derived an analytical solution for the vibration of a clamped pMUT with multiple electrodes with in-plane residual stress using the Green’s function approach [10], and generalized the model to develop an equivalent circuit model [11].

In this work we develop a circuit model representation of a clamped spherically curved pMUT in order to better understand the physics of their performances.

II. THEORY AND CIRCUIT MODEL

Previously we have solved the time harmonic deflection equation of a clamped spherical piezoelectric shell in elastic deformation analytically [6]. Figure 1 shows a 3D schematic of a curved pMUT, which consists of a piezoelectric material sandwiched between two electrodes with a backside etch-through hole to release the diaphragm [7].

![Fig. 1. 3D schematic of a curved pMUT with nominal radius of $r$ and radius of curvature of $R_c$.](image)

In order to make an equivalent circuit model for this type of transducers, the volumetric displacement and the electrical charge are determined first with respect to the input voltage and the acoustic pressure. Afterwards, the lumped circuit parameters including electrical capacitance, electromechanical transduction ratio, and mechanical admittance are extracted.

A. Volumetric Displacement

The volumetric displacement is the amount of the volume that the transducer under vibration sweeps from its static...
equilibrium position to its maximum. For a spherical shell the volumetric displacement can be calculated as:

\[ \bar{v} = R^2 \int_0^2 \int_0^\phi w(\phi) \sin \phi d\phi d\Theta \]  

(1)

where \( R \) is the radius of curvature, \( w(\phi) \) is the radial displacement of each point on the middle surface of the transducer, and \( \phi_0 \) is the azimuthal angle at the edge of the spherical shell as shown in Fig. 2.

![Fig. 2. A 2D schematic of the axisymmetric curved pMUT with clamped boundary condition, in form of a spherical shell of center \( O \) and radius \( R_c \). The radial and tangential displacements at a point \( B \) with an angular position \( \phi \) from the shell axis are denoted as \( w(\phi) \) and \( u_\phi(\phi) \), respectively. The apex point is denoted as \( A \).]

### Table I. List of the Parameters and Functions

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b(\omega) )</td>
<td>( (x + 1)[\csc \phi_0 P_{s,1}(\cos \phi_0) - \cot \phi_0 P_1(\cos \phi_0)] )</td>
</tr>
<tr>
<td>( f(x) )</td>
<td>( (x + 1)[\csc \phi_0 P_{s,1}(\cos \phi_0) - \cot \phi_0 P_1(\cos \phi_0)] )</td>
</tr>
<tr>
<td>( g(x) )</td>
<td>( \left[ \frac{(x - 1)}{2} / (1 + v) \right] \left[ \frac{1}{h} \right] + 1 )</td>
</tr>
<tr>
<td>( l_u )</td>
<td>( \left( \alpha + \frac{1}{4} \right)^2 \frac{1}{2} ) ( \alpha = 1, 2, 3 )</td>
</tr>
<tr>
<td>( a_{ij} )</td>
<td>( P_{ij}(\cos \phi_0) )</td>
</tr>
<tr>
<td>( a_{ij} )</td>
<td>( f(l_i) )</td>
</tr>
<tr>
<td>( a_{ij} )</td>
<td>( g(\lambda_i) f(l_i) )</td>
</tr>
<tr>
<td>( m_1 )</td>
<td>( (a_{23} a_{33} - a_{33} a_{22}) )</td>
</tr>
<tr>
<td>( m_2 )</td>
<td>( (a_{33} a_{23} - a_{33} a_{22}) )</td>
</tr>
<tr>
<td>( m_3 )</td>
<td>( (a_{33} a_{23} - a_{33} a_{22}) )</td>
</tr>
<tr>
<td>( n )</td>
<td>( a_{11} m_1 - a_{21} m_2 + a_{31} m_3 )</td>
</tr>
<tr>
<td>( A_j )</td>
<td>( (-1)^j m_j / n )</td>
</tr>
</tbody>
</table>

### Table II. List of the Lumped Element Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H(l_u) )</td>
<td>( \left[ P_{s,1}(\cos \phi_0) - P_{s,1}(\cos \phi_0) \right] / \left[ 1 + 2l_u \right] )</td>
</tr>
<tr>
<td>( k^2 )</td>
<td>( 2Y_0(d_{11})^2 / ((1 - v)\epsilon_r) )</td>
</tr>
<tr>
<td>( X(\omega) )</td>
<td>( \left[ (1 - \cos \phi_0) + A_i H(l_i) + A_i H(l_i) + A_i H(l_i) \right] b(\omega) )</td>
</tr>
<tr>
<td>( Y_m )</td>
<td>( 2\pi R_c^2 X(\omega) )</td>
</tr>
<tr>
<td>( b_x = b_{xx} = b_{xz} )</td>
<td>( \left[ 4\pi Y_{d,1}R_c / (1 - v) \right] X(\omega) )</td>
</tr>
<tr>
<td>( C_{xx} )</td>
<td>( \left[ (8\pi Y_{d,1}R_c^2) / (1 - v)^2 \right] X(\omega) )</td>
</tr>
<tr>
<td>( C_0 )</td>
<td>( 2\pi (1 - \cos \phi_0) R_c^2 \epsilon_r (1 - k^2) / h )</td>
</tr>
</tbody>
</table>

In our previous work, the magnitude of radial displacement of a clamped spherical piezoelectric shell has been derived explicitly [6]. The equation is rearranged here:

\[ w^* = \left[ 1 + A_i P_{i}(\cos \phi_0) + A_i P_{i}(\cos \phi_0) + A_i P_{i}(\cos \phi_0) \right] \times \left[ p_r + \frac{2Y_{d,1}R_c}{(1 - v)\epsilon_r} \right] b(\omega) \]

(2)

where \( p_r \) is the acoustic pressure, \( V_r \) is the electrical voltage between the two electrodes, \( d_{11} \) is the transverse piezoelectric strain constant, \( Y_0 \) is the Young’s modulus, and \( \nu \) is the Poisson’s ratio, \( h \) is the diaphragm thickness, \( P_i \) is the Legendre function of the first kind of order \( i \), and \( \lambda_i \) are the solutions of a characteristic function derived in [6]. \( A_i \) and \( b(\omega) \) are functions of the operation frequency, the material properties, and the geometry of the transducer as listed in Table I along with other expressions.

By integrating (2) over the surface area of the shell using (1), the total volumetric displacement can be derived as functions of \( p_r \) and \( V_r \):

\[ \bar{v}^* = Y_m p_r + b_{xx} V_r \]

(3)

where \( Y_m \) is the mechanical admittance and \( b_{xx} \) is the electrical to mechanical transduction coefficients expressed in Table II.

### B. Electrical Charge

The electric displacement field in the radial direction for a spherical piezoelectric shell can be written in the following form:

\[ D_r = d_{11}(\sigma_{th} + \sigma_{ph}) + \epsilon_r E_r \]

(4)

where \( \sigma_{th} \) and \( \sigma_{ph} \) are the transverse stresses in \( \theta \) and \( \phi \) directions respectively, \( \epsilon_r \) is the permittivity, and \( E_r \) is the applied electric field in the radial direction. By expressing \( \sigma_{th} \) and \( \sigma_{ph} \) in terms of the strains using the piezoelectric constitutive equations, followed by the substitution of the strains by the displacements in spherical coordinate system, the radial electric displacement can be derived as:

\[ D_r = \frac{Y_{d,1}}{(1 - v)R_c} \left[ \cot \phi_0 + \frac{\partial u}{\partial \phi} + 2w \right] + \epsilon_r (1 - k^2) E_r \]

(5)

In order to calculate the total electric charge, \( D_r \) must be integrated over the surface area of the transducer (i.e. the spherical shell).
\[ Q = \oint D_r \, dA \]. By integrating (5), the second system equation can be derived:

\[ Q = b_{em} p_r + C_{em} V_r + C_0 V_r \]  

(6)

where \( b_{em} \) is the mechanical to electrical transduction coefficient, \( C_{em} \) is the induced capacitance due to mechanical motion, and \( C_0 \) is the blocked capacitance as shown in Table II. It is important to note that \( b_{me} \) and \( b_{em} \) are equal, thus, the reciprocity of the system is verified and an equivalent circuit representation of such transducer is appropriate.

C. Equivalent Electrical Circuit

Having the system equations, (3) and (6), and explicit expressions for the lumped element parameters, the equivalent circuit model for the transducer can be derived to relate the electrical, mechanical, and acoustical domains [11]. Any of the circuit models shown in Fig. 3 can serve as the equivalent circuit of the transducer. In the circuit representation \( Z_e \) is the electrical feedthrough, \( Z_m \) is the mechanical impedance, \( Z_a \) is the acoustical load, and \( \eta \) is the electromechanical transduction ratio defined as:

\[ \begin{align*}
Z_e &= \frac{1}{j \omega C_0}, & Z_m &= \frac{1}{j \omega Y_m} C_m, & \text{and} & \eta &= \frac{b_m}{Y_m} \\
1: \eta
\end{align*} \]  

(7)

Fig. 3. three equivalent circuit models of the transducers.

III. RESULTS AND DISCUSSIONS

The theoretical and the simulation results are compared by using a specific example: an air-backed, 2-\( \mu \)m thick Aluminum Nitride curved pMUT with 70 \( \mu \)m in nominal radius, 720 \( \mu \)m in radius of curvature, in different front side media. COMSOL (acoustic-piezoelectric interaction) is used to simulate the frequency response of the transducer, while the theoretical analysis is coded in MATLAB. Figure 4 shows the volumetric displacement frequency responses in air and water with good consistency between the theory and simulation.

One of the essential advantages of MEMS ultrasonic transducers is their low mechanical impedance (i.e. \( Z_m \)), which can be matched well to the medium acoustic impedance, hence, eliminates the necessity of using a matching layer. From Fig. 4 we can see that the 3-dB bandwidth in water for this prototype transducer is 150 and 200 kHz while resonant frequency is predicted to be 1.24 and 1.32 MHz in theory and simulation, respectively, which is close to 94% in agreements.

Fig. 4. Volumetric displacement responses of a 2-\( \mu \)m thick AlN pMUT with 70\( \mu \)m in nominal radius and 720\( \mu \)m in radius of curvature in air and water.

Fig. 5. Comparison of the input impedance of the transducer in air (a, top) and water (b, bottom) between theory and simulation for the same device in Fig 4.

Figures 5(a) and (b) show the comparison between the theory and the simulation results of input impedance in air and water, respectively. It can be seen that the theory and the simulation
results have good consistency in terms of the trend, value, and
prediction of the resonant and anti-resonant frequencies.

Electromechanical coupling factor ($k_{eff}^2$) indicates the
conversion efficiency of the electrical energy into the
mechanical energy. For piezoelectric transducers, the
maximum possible value depends on the material properties
and the operation mode. In the case of curved pMUTs, the
expression for maximum $k_{eff}^2$ is the same as $k^2$ shown in Table
II and its value for a clamped AlN diaphragm operating in
flexural mode is ~5.5%. The actual value of $k_{eff}^2$ can be
calculated from the input impedance versus the frequency of
the transducer operating in vacuum using:

$$k_{eff}^2 = \frac{f_r^2 - f_a^2}{f_r^2}$$  \hspace{1cm} (8)

where $f_r$ and $f_a$ are the resonant and anti-resonant frequencies
respectively. Figure 6 shows the electromechanical coupling
factor versus the radius of curvature for curved AlN pMUTs
with identical thicknesses (2 µm) and nominal radii (70 µm). It
is observed in Fig. 6 that both theory and simulation results
have similar trends: $k_{eff}^2$ will first increase by increasing the
radius of curvature, reaches a climax, and finally descends.

![Graph](image)

Fig. 6. Electromechanical coupling factor versus the radius of curvature of a
2-µm thick AlN pMUT with 70 µm nominal radius.

The maximum electromechanical coupling factor for this
specific curved pMUT is predicted to be 3.3% which shows
curved pMUT can potentially reach 60% of material limit – the
value of AlN is ~5.5%. This result shows that curved pMUT is
much more responsive than conventional planar AlN pMUT
(maximum electromechanical coupling factor is predicted to be
0.5% [12]). The corresponding optimum value for the radius of
curvature is 520 µm in this case. Using such analysis, we can
predict the optimum design for curved pMUTs, based on the
desired operation frequency and application.

IV. CONCLUSION

We have derived an equivalent circuit model for clamped
curved piezoelectric shells including all lumped element
parameters. This theoretical model has been validated by
numerical simulations using CMOSOL. The results show that
curved pMUTs can reach 60% of the theoretical piezoelectric
material limit which is much more responsive than the
maximum limit of the state-of-the-art flat pMUTs.