Mutual 3:1 subharmonic synchronization in a micromachined silicon disk resonator

Parsa Taheri-Tehrani,1,a) Andrea Guerrieri,2,b) Martial Defoort,1 Attilio Frangi,2 and David A. Horsley1,a)

1Department of Mechanical and Aerospace Engineering, University of California, Davis, California 95616, USA
2Department of Civil and Environmental Engineering, Politecnico di Milano, 20133 Milano, Italy

(Received 21 July 2017; accepted 21 October 2017; published online 1 November 2017)

We demonstrate synchronization between two intrinsically coupled oscillators that are created from two distinct vibration modes of a single micromachined disk resonator. The modes have a 3:1 subharmonic frequency relationship and cubic, non-dissipative electromechanical coupling between the modes enables their two frequencies to synchronize. Our experimental implementation allows the frequency of the lower frequency oscillator to be independently controlled from that of the higher frequency oscillator, enabling study of the synchronization dynamics. We find close quantitative agreement between the experimental behavior and an analytical coupled-oscillator model as a function of the energy in the two oscillators. We demonstrate that the synchronization range increases when the lower frequency oscillator is strongly driven and when the higher frequency oscillator is weakly driven. This result suggests that synchronization can be applied to the frequency-selective detection of weak signals and other mechanical signal processing functions.

Published by AIP Publishing. https://doi.org/10.1063/1.4997195

Phase synchronization is a well-known phenomenon used in several scientific fields.1–4 The first documented reports of the synchronization phenomenon date back to the seventeenth century when Huygens5 observed synchronization between two self-sustained, almost identical, ship navigation clocks whose motion was coupled through a common suspension beam. The most basic form of synchronization is based on unidirectional coupling and is also known as injection locking.6–8 The frequency of a first oscillator, \( f_1 \), is caused to synchronize (lock) to the frequency of a second reference oscillator, \( f_2 \), by injecting the second oscillator’s output into the first oscillator. The range of frequencies over which this locking can occur is called the synchronization range, and this range increases with the amplitude of the reference oscillator and with certain types of amplitude-dependent nonlinearities in the first oscillator.9,10 Synchronization can also occur when the two frequencies have a harmonic relationship, \( f_2 = nf_1 \), where \( n \) is an integer, a phenomenon known as subharmonic synchronization.11–14 Distinct from injection locking, mutual synchronization and mutual subharmonic synchronization occur when there is mutual, bidirectional coupling between the two oscillators.15–20 In oscillators based on micro- and nano-electromechanical systems (MEMS and NEMS) resonators, mutual synchronization is of interest because, assuming constant strain-energy density, oscillator phase noise increases as the physical volume of a resonator decreases. As a result, NEMS and MEMS-based oscillators tend to have greater phase noise than their macroscopic equivalents. However, it has been shown that phase noise can be reduced below the individual-resonator level by mutually synchronizing multiple resonators.21,22 Synchronization in these NEMS and MEMS devices is most commonly demonstrated by coupling two different resonators either electronically17 or mechanically via electrostatic force,20,22 light,23,24 or flexures.25,26 The need to use multiple resonators in these demonstrations introduces a serious flaw in this approach, since the original goal of miniaturization is reduced size. Here, we consider a different approach that requires no additional device area: rather than synchronizing two resonators operating at the same vibration mode, we start with a single resonator and synchronize two different vibration modes which have a 3:1 subharmonic frequency relationship. To date, this approach has seen little attention, in part due to the difficulty of obtaining close 3:1 frequency-matching between the two modes. Antonio et al.27 solved the frequency-matching problem by exploiting a resonator nonlinearity that results in amplitude-frequency (A-f) dependence—the first vibration mode’s amplitude was adjusted until the desired 3:1 frequency relationship with the second mode was achieved. This solution has the disadvantage that the effect of resonator amplitude on the synchronization range cannot be studied independently since amplitude and frequency are coupled. To overcome this limitation, we study synchronization using a micromachined disk resonator (MDR) originally designed to operate as a mode-matched gyroscope.28 Mode-shape specific frequency-tuning electrodes enable the resonant frequency of the first mode to be voltage-controlled, independent of the frequency of the second mode. This capability enables us to study the synchronization range as a function of the frequency and amplitude of the two modes. We observe that the synchronization range increases as the first mode is driven to larger amplitudes but, surprisingly, the range decreases as the amplitude of the second, higher-frequency mode is increased. This result suggests...
that mutual synchronization may have application to the detection of weak signals around the frequency of the second mode, as any perturbation to this mode’s amplitude would result in loss of synchronization and a corresponding change in oscillation frequency.

The MDR is a 40 μm thick, 600 μm diameter, single-crystal Si (Silicon) disk that is perforated by concentric annular slots and supported by a central anchor, shown in Fig. 1(a). The fabrication process, described elsewhere, uses a Si (Silicon) epitaxy reactor to vacuum seal the released resonator in an inert gas environment at a pressure of 1 Pa, a process that results in high quality factor \(Q\) of the resonance modes and a high degree of frequency stability, due to the cleanliness of the resonator surface. The MDR is surrounded by parallel-plate electrodes that enable electrostatic transduction and frequency tuning of the resonance modes. Two separate electrode pairs are used to drive and sense the motion of the two resonance modes. Phase-locked loops (PLLs) implemented in a digital lock-in-amplifier (LIA, Zurich Instruments HF2LI) are used to operate the two resonance modes as self-sustained oscillators (e.g., van der Pol self-sustained oscillators, \(^{30,31}\) by locking the oscillation phase at –90° relative to the drive voltage.\(^{32}\)

Figure 1(b) shows the measured frequency sweep of the two resonance modes of interest along with their mode shapes simulated via finite element model ([FEM], Comsol Multiphysics). As shown in the figure, each mode shape is described by \(u(\theta) = u_n \cos(n\theta + \phi)\), where \(u_n\) is the maximum radial displacement, \(\theta\) is the angular coordinate, \(n\) is an integer, and \(\phi\) is the rotation of the first antinode relative to the electrode coordinates. Note that, while a perfect disk exhibits degenerate \(\cos(n\theta), \sin(n\theta)\) mode pairs, fabrication imperfections split the frequencies of these pairs such that they are separated by more than the 100 Hz measured frequency range.\(^{28}\) In this letter, the first mode of interest is an \(n = 3\) mode (referred to as \(3\) mode) and has a resonance frequency of \(f_1 = 278\) kHz and a quality factor of \(Q_1 = 15,000\). The second mode is an \(n = 9\) mode (referred to as \(9\) mode) with \(Q_2 = 60,000\) and was selected because it has a resonance frequency that is nearly a harmonic of the first frequency \((f_2 \approx 3 f_1)\). As a result of cubic stiffness-softening introduced by the parallel-plate electrodes,\(^{33}\) the two modes exhibit Duffing nonlinear amplitude-frequency (A-f) dependence.\(^{32,34}\)

To effectively study the synchronization range and the interactions of the two oscillators within this range, it is essential to be able to change the frequency of one oscillator independent from that of the other oscillator. Figure 1(c) presents measurements to show that the frequency of the \(3\) \(\theta\) oscillator can be controlled independently from the frequency of the \(9\) \(\theta\) oscillator using voltage applied to a tuning electrode. When the tuning voltage is increased, the voltage difference between the disk (which is biased at 16.5 V) and the tuning electrode decreases, decreasing this electrode’s electrostatic stiffness and increasing \(f_1\). The increase in frequency is quadratic because the electrostatic stiffness-constant is proportional to the square of the voltage difference.\(^{35}\) The same experiment illustrates that \(f_2\) remains unchanged as the tuning voltage is varied. This independence in tuning of the frequency allows the interaction of the two modes to be studied as \(f_1\) is varied in the neighborhood of \(f_2/3\) in the synchronized and unsynchronized states. Independent tuning is possible for two main reasons: first, the selected tuning electrode overlaps both nodes and anti-nodes of the \(9\) mode shape, resulting in weak coupling to this mode;\(^{35}\) and second, the stiffness of the \(9\) mode is much higher than that of the \(3\) mode, and is much greater than the electrostatic stiffness of the tuning electrode.

Figure 2 illustrates the measured synchronization between the \(3\) \(\theta\) and \(9\) \(\theta\) modes as the \(3\) \(\theta\) mode’s frequency is tuned over a \(1\) V range. In these experiments, the \(9\) \(\theta\) mode (at frequency \(f_2\)) is weakly excited with \(35\) mV and the \(3\) \(\theta\) mode (at frequency \(f_1\)) is strongly driven at two different amplitudes to demonstrate the dependence of the synchronization range on the oscillator’s amplitude. As the tuning voltage is increased, \(f_1\) increases and approaches \(f_2/3\) from below (upwards frequency sweep). As \(f_1\) approaches \(f_2/3\), the amplitude of quasi-periodic oscillation with a frequency of \(f_1 - f_2/3\) increases until the two frequencies collapse on each other. At this stage, the \(9\) \(\theta\) mode’s oscillation frequency, despite being independent from the tuning voltage, will follow \(3f_1\) due to synchronization. As the tuning voltage is increased further, the modes eventually exit the synchronized state and quasi-periodic oscillations in both modes reappear. The same experiment was conducted by decreasing the tuning voltage such that \(f_1\) approaches \(f_2/3\) from above (downwards frequency sweep). A similar behavior is observed in this experiment as well, with a difference that, when the \(3\)
applied to the modes. The stiffness terms shown in Fig. 1, both modes exhibit Duffing nonlinearities essential to make subharmonic synchronization possible. These nonlinearities create internal resonance between the two resonance modes are coupled through linear and nonlinearities and this asymmetry has been previously reported in injection-locked oscillators and mutual synchronization. Finally, the surprising result that increasing the energy of the second mode to the 9th mode results in a decrease in the synchronization range compared with the numerical solution of the coupled oscillators, we introduce a model which explains this behavior. The two resonance modes are coupled through linear and nonlinear electromechanical stiffness terms. This coupling is essential to make subharmonic synchronization possible. The model includes cubic coupling terms because first, as shown in Fig. 1, both modes exhibit Duffing nonlinearities and second, in a first-order approximation, only cubic coupling nonlinearities create internal resonance between the two resonance modes. Simplified two degree-of-freedom (DOF) equations of motion can model the behavior of the resonance modes in the synchronized state

$$\begin{align*}
\ddot{x}_1 + \omega_1^2 x_1 &= k_1 x_1^3 + k_2 x_1^5 + k_3 x_1 x_2^2 + k_4 x_2^3 + \lambda_1, \\
\ddot{x}_2 + \omega_2^2 x_2 &= k_5 x_1^3 + k_6 x_1 x_2^2 + k_7 x_1 x_2^2 + k_8 x_2^3 + \lambda_2.
\end{align*}$$

(1)

In these equations, $x_1$ and $x_2$ are the modal amplitudes of the 30 and 90 resonance modes with resonant frequencies $\omega_1 = 2\pi f_1$ and $\omega_2 = 2\pi f_2$, respectively. $\Delta \omega_1 = \omega_1 / Q_1$ and $\Delta \omega_2 = \omega_2 / Q_2$ are the full width at half maximum (FWHM) bandwidths of the two modes. $\lambda_1$ and $\lambda_2$ are external forces applied to the modes. The stiffness terms $k_1$ and $k_8$ are cubic terms that result in Duffing nonlinearity in single DOF resonators, while $k_2$ through $k_7$ represent the nonlinear coupling between the two modes. Using a perturbation and multiple time-scales method with $x_1 = 1/2X^2 e^{i(\omega t + \beta_1)} + cc$, $x_2 = 1/2X^2 e^{i(\omega t + \beta_2)} + cc$, $\lambda_1 = 1/2X_1 e^{i(\Omega t + \gamma_1)} + cc$, and $\lambda_2 = 1/2X_2 e^{i(\Omega t + \gamma_2)} + cc$, where $cc$ is the complex conjugate, these equations can be solved by a first-order approximation and result in the following amplitude and phase equations of each mode:

$$\begin{align*}
\dot{X}_1 &= K_1 \sin(\gamma_1) + \delta_1 X_2 X_1^2 \sin(\gamma_2) - \frac{\Delta \omega_1}{2} X_1, \\
\dot{\theta}_1 &= -K_1 \cos(\gamma_1) - \alpha_1 X_2^2 - \beta_1 X_2^2 - \delta_1 X_1 X_2 \cos(\gamma_3), \\
\dot{X}_2 &= K_2 \sin(\gamma_2) - \delta_2 X_1^3 \sin(\gamma_3) - \frac{\Delta \omega_2}{2} X_2, \\
\dot{\theta}_2 &= -K_2 \cos(\gamma_2) - \alpha_2 X_2^2 - \beta_2 X_2^2 - \delta_2 \frac{X_1^3}{X_2} \cos(\gamma_3).
\end{align*}$$

(2)

Here, $\gamma_1$ and $\gamma_2$ are the phase-shifts of the external forces $\lambda_1$ and $\lambda_2$ which are fixed by the PLL, while $\gamma_3$ is associated with the phase difference between the oscillations of the two modes. For simplification and to provide an intuitive explanation of coupling, we have introduced four new variables: $K_1$, $\alpha$, $\beta$, and $\delta$. These variables are further explained in the supplementary material; here a concise explanation is provided. $K_1$ and $K_2$ represent the external forcing terms. The next four terms relate to amplitude-frequency dependence: $\alpha_1 X_1^2$ and $\alpha_2 X_2^2$ are the standard terms causing a shift in frequency due to Duffing nonlinearity, while $\beta_1 X_2^2$ and $\beta_2 X_2^2$ are terms causing a shift in frequency due to coupling between the two modes. Measured values for these four terms are presented in the supplementary material. Finally, $\delta_1$ and $\delta_2$ are the coupling coefficients which cause the internal resonance and consequently synchronization between the two modes. The terms including $\delta_1$ and $\delta_2$ in the equations, depend on the amplitudes of both modes, resulting in amplitude dependence of the synchronization behavior. For example, $\delta_1 X_1 X_2 \cos(\gamma_3)$ and $8\delta_2 X_1^3 X_2^2 \cos(\gamma_3)$, both depend on $X_1$ and $X_2$. The coupling term $8\delta_2 X_1^3 X_2^2 \cos(\gamma_3)$ decreases as $X_2$ (the amplitude of the 9th mode) increases; this fact results in decreased coupling from the 30 mode to the 9th mode and consequently the synchronization range decreases. On the other hand, all the coupling terms increase with the amplitude of the 30 mode ($X_1$), resulting in an increased synchronization range.

Previous studies have shown that in a single nonlinear oscillator, the synchronization range increases with the oscillator’s energy (amplitude squared). The model in Eq. (2) predicts similar behavior as the energy of the first oscillator (30 mode) is increased. More importantly, it also predicts the surprising result that increasing the energy of the second oscillator (90 mode) results in a decrease in the synchronization range. This observation is in contrast with the reported behavior of injection-locked oscillators and mutually synchronized oscillators in which the synchronization range always increases with oscillator amplitude. We have conducted measurements to experimentally prove these predictions. Figure 3 illustrates a 3D plot of the measured synchronization range compared with the numerical solution from the model of Eq. (2) for drive voltages ranging from 450 mV to 750 mV for the 30 mode and from 15 mV to 55 mV for the 90 mode. These data illustrate that the synchronization range increases with the energy in the first mode ($X_1^2$) and has an inverse relationship with the energy in the second mode ($X_2^{-2}$).
provided in the supplementary material. Two oscillators based on two distinct resonance modes of a parameter values needed to reproduce Figs. 3 and 4 are produced by the Duffing nonlinearity, $x_1^3$ and $x_2^3$. The measured amplitudes, phase, and frequency presented in Fig. 4 show the synchronization range increases with the $3\omega$ mode oscillator’s amplitude and decreases with the $9\omega$ mode oscillator’s amplitude.

To further validate the model and understand the synchronization dynamics, we compare the measured behavior of the oscillators in the synchronized state with the behavior predicted by the model. While synchronized, the frequency of the two modes will follow each other. The measured amplitude, phase, and frequency presented in Fig. 4 show the effect which is also predicted by the model, and is different from the behavior of injection-locked oscillators. Looking at the oscillator amplitudes in Fig. 4, this frequency deviation occurs at a point where the bidirectional coupling results in energy transfer between the two modes: $X_1$ diminishes and $X_2$ grows. The changing amplitudes in turn affect the frequency through the amplitude-frequency dependence produced by the Duffing nonlinearity, $x_1^3$ and $x_2^3$. The parameter values needed to reproduce Figs. 3 and 4 are provided in the supplementary material.

In conclusion, we demonstrated synchronization between two oscillators based on two distinct resonance modes of a single resonator. Using a coupled-oscillator model, we showed that this 3:1 subharmonic synchronization originates from cubic, non-dissipative electromechanical coupling. In agreement with the model, the synchronization range for subharmonic frequency entrainment increases with the energy in the higher frequency oscillator. These results illustrate the previously unexplored dynamics of subharmonic synchronization between two intrinsically coupled modes in a single micromechanical resonator, a phenomenon which may have applications in weak signal detection, mechanical signal processing, and frequency stabilization of micromechanical sensors and timing oscillators.

See supplementary material for the model and numerical analysis along with additional measurements of the MDR’s characteristics.

The authors thank Professor Thomas Kenny and his research group at Stanford University for device fabrication. This work was funded by Berkeley Sensor and Actuator (BSAC) members and DARPA Grant No. W31P4Q-11-1-0003.

---

**FIG. 3.** 3D plot illustrating the measured (spheres) and simulated (surfaces) dependence of the synchronization range on the drive voltages of both oscillators. The synchronization range increases with the $3\omega$ mode oscillator’s amplitude and decreases with the $9\omega$ mode oscillator’s amplitude.

**FIG. 4.** Synchronization: comparison of the model (red dashed lines) and the measurements (solid black lines): (a) synchronized frequency, (b) synchronized phase, (c) amplitude of the first mode, and (d) amplitude of the second mode. In the figure, $\sigma_3 = \omega_3 - 3\omega_1$ represents the frequency detuning of the two oscillators.