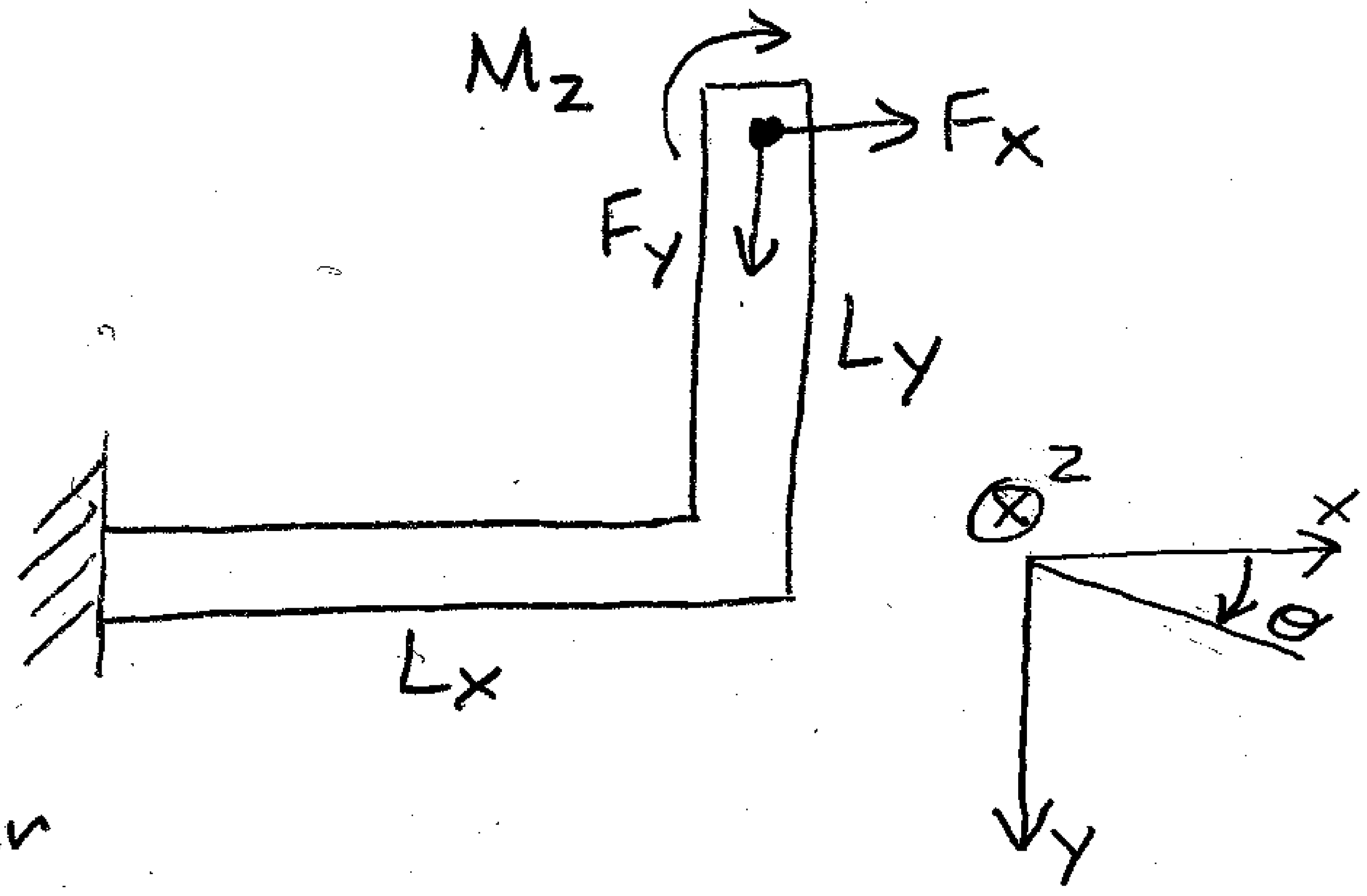
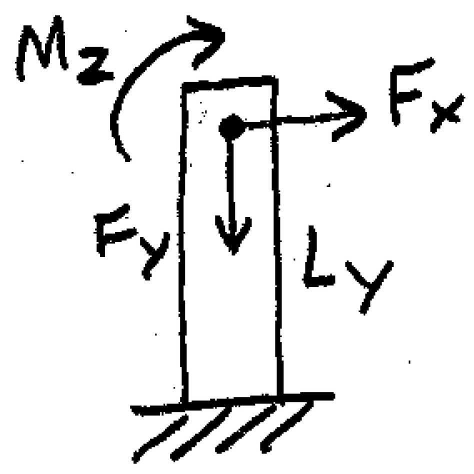
F_x, F_y and M_z result in displacements v_x, v_y, v_θ

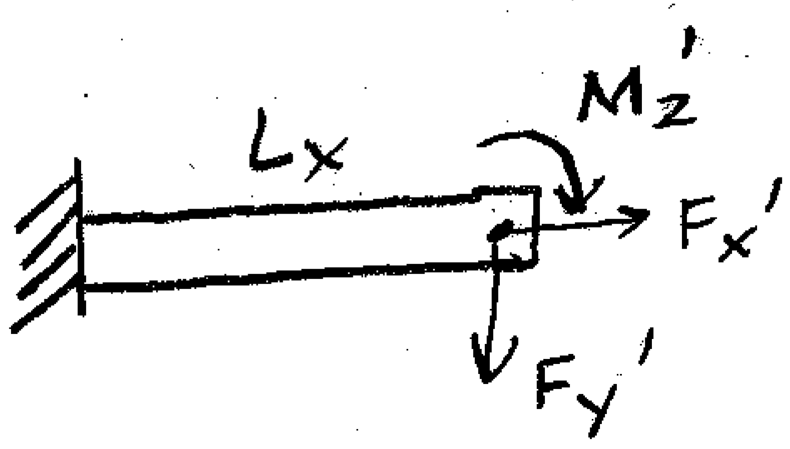


Split problem into two, and take linear combination for final answer

① v_{x1}, v_{y1}, $v_{\theta 1}$



② v_{x2}, v_{y2}, $v_{\theta 2}$

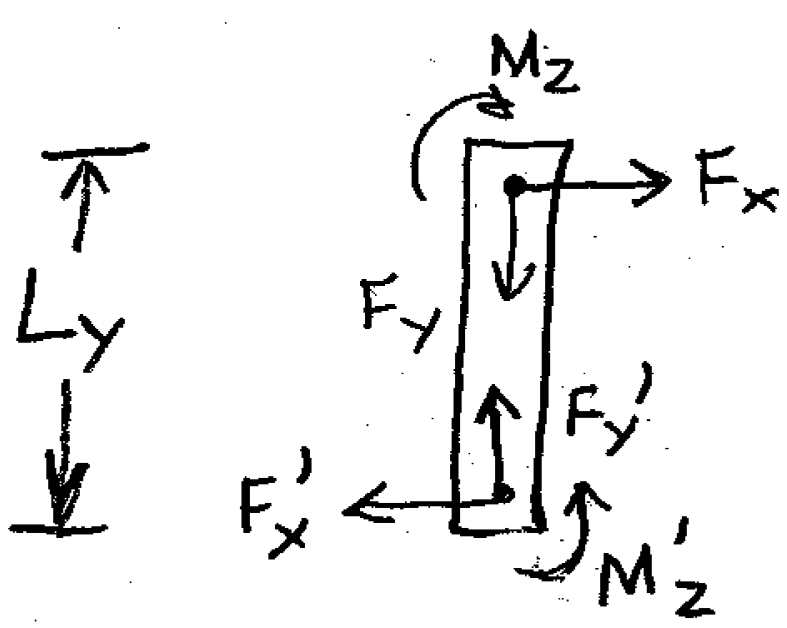


① $$v_{x1} = \frac{L_y^3}{3EI} F_x + \frac{L_y^2}{2EI} M_z$$

$$v_{y1} = 0 \quad \text{(no axial compliance)}$$

$$v_{\theta 1} = \frac{L_y^2}{2EI} F_x + \frac{L_y}{EI} M_z$$

② Find reaction forces - balance forces & moments in free body diagram



$$0 = F_x - F_x' \Rightarrow F_x' = F_x$$

$$0 = F_y - F_y' \Rightarrow F_y' = F_y$$

$$0 = M_z + F_x L_y - M_z'$$
$$\Rightarrow M_z' = M_z + F_x L_y$$

check $$0 = M_z + F_x' L_y - M_z'$$
$$\Rightarrow M_z' = M_z + F_x' L_y = M_z + F_x L_y \checkmark$$

$V_{x2} = 0$ (no axial compliance)

$$V_{y2} = \frac{L_x^3}{3EI} F_y' + \frac{L_x^2}{2EI} M_z'$$

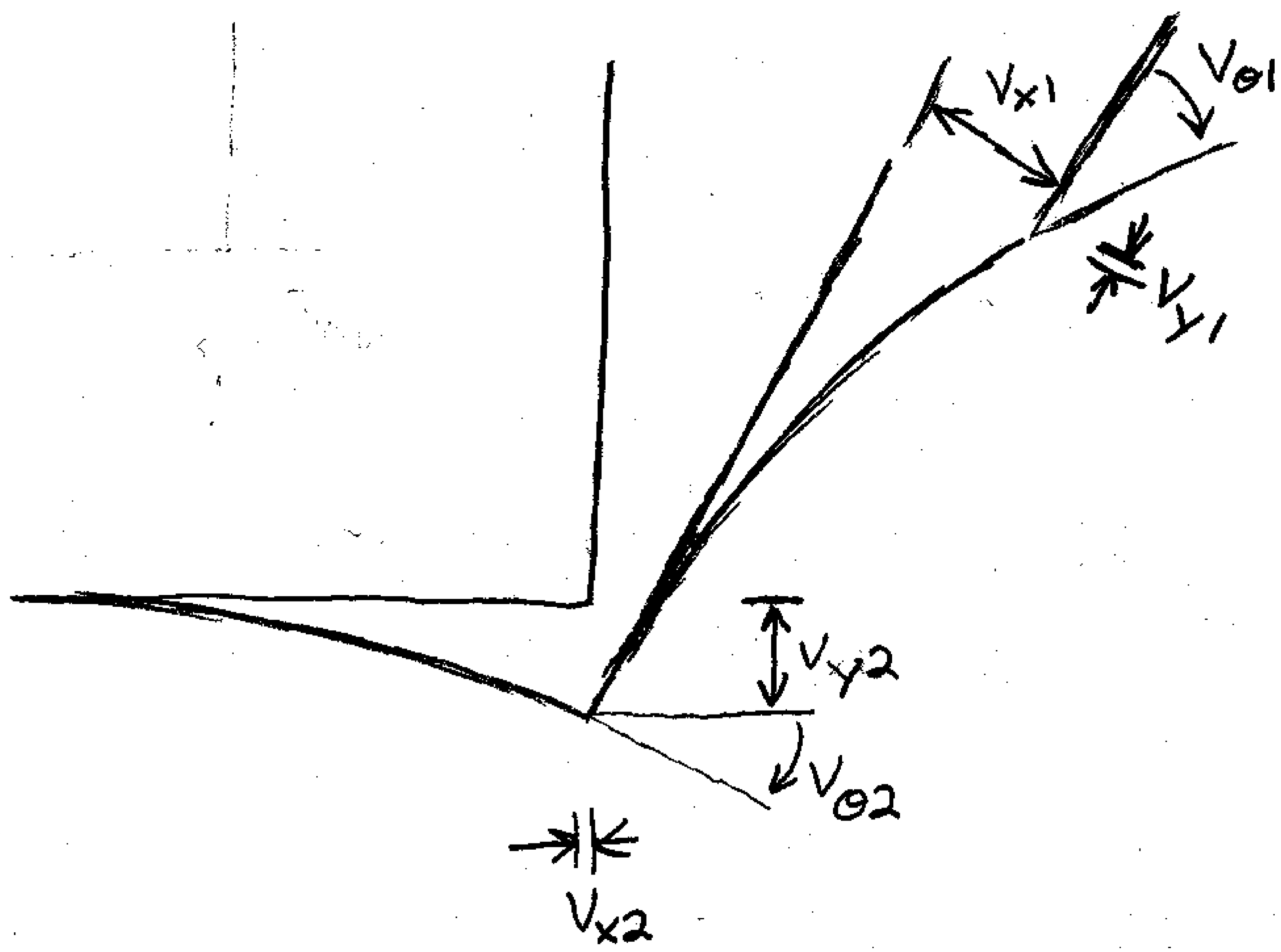
$$= \frac{L_x^3}{3EI} F_y + \frac{L_x^2}{2EI} (M_z + F_x L_y)$$

$$= \frac{L_x^2 L_y}{2EI} F_x + \frac{L_x^3}{3EI} F_y + \frac{L_x^2}{2EI} M_z$$

$$V_{\theta 2} = \frac{L_x^2}{2EI} F_y' + \frac{L_x}{EI} M_z'$$

$$= \frac{L_x^2}{2EI} F_y + \frac{L_x}{EI} (M_z + F_x L_y)$$

$$= \frac{L_x L_y}{EI} F_x + \frac{L_x^2}{2EI} F_y + \frac{L_x}{EI} M_z$$



$$V_x = V_{x2} + L_y \cancelto{V_{\theta 2}}{\sin V_{\theta 2}} + V_{x1} \cancelto{1}{\cos V_{\theta 2}} + V_{y1} \cancelto{V_{\theta 2}}{\sin V_{\theta 2}}$$

$$= \cancelto{0}{V_{x2}} + L_y V_{\theta 2} + V_{x1} + \cancelto{0}{V_{y1}} V_{\theta 2}$$

$$V_y = V_{y2} + L_y (1 - \cancelto{1}{\cos V_{\theta 2}}) + V_{y1} \cancelto{1}{\cos V_{\theta 2}} + V_{x1} \cancelto{V_{\theta 2}}{\sin V_{\theta 2}}$$

$$= V_{y2} + \cancelto{0}{V_{y1}} + V_{x1} V_{\theta 2}$$

$$V_\theta = V_{\theta 2} + V_{\theta 1}$$

This is the one non-zero cross term. It's small, so we'll ignore it.

$$V_x = L_y\left(\frac{L_xL_y}{EI}F_x + \frac{L_x^2}{2EI}F_y + \frac{L_x}{EI}M_z\right) + \frac{L_y^3}{3EI}F_x + \frac{L_y^2}{2EI}M_z$$

$$= \left(\frac{L_xL_y^2}{EI} + \frac{L_y^3}{3EI}\right)F_x + \left(\frac{L_x^2L_y}{2EI}\right)F_y + \left(\frac{L_xL_y}{EI} + \frac{L_y^2}{2EI}\right)M_z$$

$$V_y = \left(\frac{L_x^2L_y}{2EI}\right)F_x + \left(\frac{L_x^3}{3EI}\right)F_y + \left(\frac{L_x^2}{2EI}\right)M_z$$

$$V_\theta = \left(\frac{L_xL_y}{EI} + \frac{L_y^2}{2EI}\right)F_x + \left(\frac{L_x^2}{2EI}\right)F_y + \left(\frac{L_x}{EI} + \frac{L_y}{EI}\right)M_z$$

$$\begin{bmatrix} V_x \\ V_y \\ V_\theta \end{bmatrix} = \begin{bmatrix} \frac{3L_xL_y^2 + L_y^3}{3EI} & \frac{L_x^2L_y}{2EI} & \frac{2L_xL_y + L_y^2}{2EI} \\ \frac{L_x^2L_y}{2EI} & \frac{L_x^3}{3EI} & \frac{L_x^2}{2EI} \\ \frac{2L_xL_y + L_y^2}{2EI} & \frac{L_x^2}{2EI} & \frac{L_x + L_y}{EI} \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ M_z \end{bmatrix}$$

if $L_x = L_y \equiv L$, then compliance matrix is:

$$\frac{1}{EI}\begin{bmatrix} \frac{4}{3}L^3 & \frac{1}{2}L^3 & \frac{3}{2}L^2 \\ \frac{1}{2}L^3 & \frac{1}{3}L^3 & \frac{1}{2}L^2 \\ \frac{3}{2}L^2 & \frac{1}{2}L^2 & 2L \end{bmatrix}$$