CHAPTER 6

Beam theory blah

Synopsis

Moment/curvature relationship $\kappa = \frac{1}{\rho} = \frac{M}{EI}, I = \frac{a^3 b}{12}$

Bending under end loading $y(x) = \frac{x^2}{6EI}(x - 3L)F + \frac{x^2}{2EI}M$

$$y(L) = \frac{L^3}{3EI}F + \frac{L^2}{2EI}M$$

Axial compression: $K_x = \frac{L}{EA}F$

Torsional deflection $\theta = \frac{ML}{JG}$

Matrix representation of end deflection vs. applied force/moment

Linear beam theory

Solving for the 3D deformation of an arbitrary object under arbitrary loading conditions is the domain of the theory of elasticity, and in general a nasty problem. We reduce the problem to a single dimension, where all quantities on the beam vary only as functions of a single variable such as arc length. This is the fundamental assumption of beam theory, which is tractable for hand analysis, and very often just as accurate for MEMS problems as the full three dimensional theory of elasticity.

A further assumption is that the deflections are "small". Let us start by assuming that the deflections are infinitesimal, and then see what the limits to this linear beam theory are in the next chapter.

We will also assume that the cross-sections of the beams vary smoothly in shape along the length of the beam, and that the cross-sectional dimensions are small compared to the length of the beam.

The goal of this chapter will be to develop the relationship between forces and moments applied at the ends of a beam and the resulting deflections. Since this is a linear theory, the resulting relationship will take the form of a matrix.

Bending

Find yourself a big rubber eraser and draw a cartesian grid on one side. Now bend the eraser by pushing your thumbs up in the middle and pulling down on the ends. You should see something like what's in Figure xxx. Your fingers are doing a decent job of applying a uniform moment along the length of the eraser, and it is doing its best to bend to a constant radius of curvature.

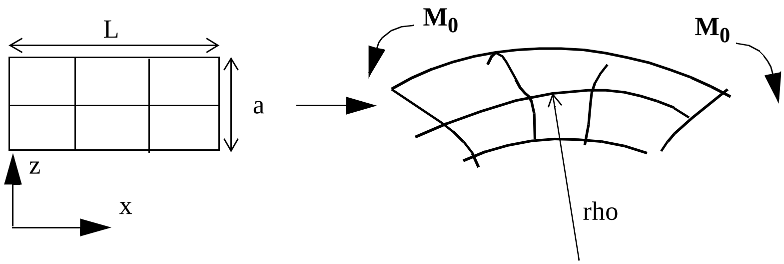

FIGURE 23. **Deformation of a cartesian grid under a uniform torque.**

A horizontal line midway between the top and the bottom of the eraser is known as the neutral axis because it will stay the same length before and after the bending, whereas lines above the midpoint will get longer, and lines below the midpoint will get shorter.

We define a coordinate system on the undeformed beam (eraser) where x is the distance along the length of the beam and z is the vertical distance above the neutral axis of the beam. If we take the thickness of the beam to be a, then we see that z varies from -a/2 to a/2.

From geometrical arguments we can show that the strain as a function of position along the beam is

$$\varepsilon(x, y, z) = \varepsilon(x, z) = \frac{z}{\rho(x)} \tag{EQ 27}$$

where we are allowing the radius to vary as a function of position along the length of the beam.

Recalling that the stress and strain are related by the Young's modulus, we can write an expression for the stress (in the x direction) as a function of position along the beam

$$\sigma_{xx}(x, y, z) = E\frac{z}{\rho(x)} \tag{EQ 28}$$

Integrating over the cross-section of the beam we see that

$$M(x) = \int_{-\frac{a}{2}}^{\frac{a}{2}} z\sigma_{xx} b\,dz = \int_{-\frac{a}{2}}^{\frac{a}{2}} zE\frac{z}{\rho(x)} b\,dz = \frac{E}{\rho(x)} b \frac{z^3}{3}\Bigg|_{-\frac{a}{2}}^{\frac{a}{2}} = \frac{Ea^3 b}{12\rho(x)} \qquad \textbf{(EQ 29)}$$

Which is the moment/curvature relationship for beams, and is usually written

$$\frac{1}{\rho(x)} = \frac{M(x)}{EI(x)} \qquad \textbf{(EQ 30)}$$

where ρ is the radius of curvature, M is the moment, and

$$I = \frac{a^3 b}{12} \qquad \textbf{(EQ 31)}$$

is the moment of inertia of the cross-section. The cross-sectional geometry can vary slowly as a function of position along the beam, making I a function of x. In principle, E can also be a function of x. Discontinuities or rapid changes in the cross section require more detailed modeling.

The curvature of the beam is approximately the second derivate of displacement y with respect to a fixed coordinate system x

$$\kappa = \frac{\frac{d^2 y}{dx^2}}{\left(1 + \frac{dy}{dx}^{3/2}\right)^{1/2}} \approx \frac{d^2 y}{dx^2} \qquad \textbf{(EQ 32)}$$

from which we see that

$$\frac{d^2 y}{dx^2} \approx \frac{M(x)}{EI} \qquad \textbf{(EQ 33)}$$

Example: Calculate the shape of a cantilevered beam with a pure moment applied at the free end of the beam.

Soln: The cantilevered beam implies a boundary condition of

$$y(0) = 0, \frac{dy}{dx}(0) = 0 \qquad \textbf{(EQ 34)}$$

We can calculate the moment at any point along the beam using a freebody diagram where we break the beam at a location x along its length. In order to maintain static equilibrium we must have the sum of all forces and moments on a body equal to zero. This implies that across a virtual break at location x, we would need to apply

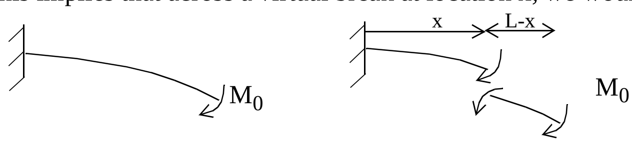

FIGURE 24. Freebody diagram for transmission of moment along the beam at location x.

-M0 on the left end of end of the end body, which implies that a moment M0 must be applied on the right end of the remaining cantilever. So

$$M(x) = M_0 \qquad \textbf{(EQ 35)}$$

Now we have an ordinary differential equation with a boundary condition. We can integrate the ODE directly, yielding

$$\frac{dy}{dx} = \frac{M_0}{EI}x + C_1$$

$$y(x) = \frac{M_0}{EI}\frac{x^2}{2} + C_1 x + C_2$$

Subsituting the boundary conditions shows that the constants of integration are zero. Note that the actual shape of the deflected beam will be a section of a circle (a curve with a constant radius of curvature). The expression that we derived is an accurate approximation to a circle for x much less than ρ. The error between the expression that we calculated and the true shape is due to the approximation that the radius of curvature was equal to the reciprocal of the second derivative.

Often we are just interested in the deflection of the endpoint of the beam, which is

$$y(L) = \frac{M}{EI}\frac{L^2}{2}; \theta(L) = \frac{dy}{dx}(L) = \frac{M}{EI}L$$

which gives us two spring constants

$$K_{y\theta} = \frac{L^2}{2EI}, K_{\theta\theta} = \frac{L}{EI}$$

Example: Consider an isotropic beam of length L and cross section a by b. Compare the angular deflection when a moment is applied axially or transversely on the beam.

Soln: Since we are in the linear region we can compare deflections simply by looking at a ratio of the two spring constants.

Example: A cantilevered beam has a rigid body bolted on the free end. A force transverse to the end of the beam is applied on the body at a distance r from the end of the cantilever. Find the value for r which minimizes the angular deflection of the tip of the cantilever, and calculate the resulting linear stiffness.

Soln: $r = -L/2$, $K = k_0 4$

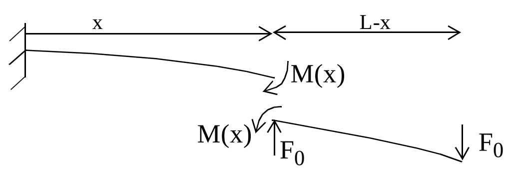

FIGURE 25. Freebody diagram for transmission of moment along a beam with force loading.

etc

Ex: apply load st. the moment is zero at the center of the beam

Ex: deflection of a beam under its own weight

Ex: residual stress induced bending

Caveats: only works for small deflections

Moments of common cross-sections

Example: moment of an I-beam

Bi-metal and composite beams

Anticlastic curvature and bending of plates

The Poisson's ratio tells us that if the top of the beam is in tension in the x direction then there will be local contraction in the y direction (orthogonal to the axis and direction of bending). Similarly, if the bottom of the beam is in compression, then there will be an expansion in the y direction. This results in a moment along the width of the beam which will tend to bend it upward.

(EQ 36)

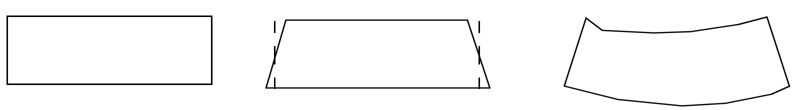

FIGURE 26. Anticlastic curvature.

Beams which are wide compared to their thickness can not expand and contract laterally as much as narrow beams. This results in an increased equivalent stiffness

$$E_{plate} = \frac{E}{1 - \nu^2} \tag{EQ 37}$$

Torsion

Consider a cylindrical bar with an equal and opposite axial torque of magnitude M applied to the two ends, resulting in a twisting of the bar by an angle θ. Looking at any cross-section along the length of the bar we see from geometric arguments that the shear strain at a distance r from the central axis is

$$\gamma = \frac{\theta r}{L} \tag{EQ 38}$$

We can calculate the moment required to generate that strain by integrating the moment induced by the shear stress over the surface of the cross-section

$$M = \int_0^R \int_0^{2\pi} r \cdot dF = \int_0^R \int_0^{2\pi} r \cdot \tau dA = \int_0^R \int_0^{2\pi} r \cdot (G\gamma r) dr d\theta \tag{EQ 39}$$

$$M = \frac{2\pi G R^4}{L}\theta = \frac{JG}{L}\theta \tag{EQ 40}$$

where

$$J = \frac{\pi R^4}{2} \tag{EQ 41}$$

J is known as the polar moment of inertia, and the formula given above is true for circular cross sections. For rectangular cross sections the formula is

$$K_\theta = \frac{KG}{L} \tag{EQ 42}$$

where for a section of dimension 2a by 2b

$$K_{rect} = ab^3\left[\frac{16}{3} - 3.36\frac{b}{a}\left(1 - \frac{b^4}{12a^4}\right)\right] \tag{EQ 43}$$

for $a \geq b$ (from Roark). In the case of a square cross-section of dimension 2a this reduces to $K = 2.25a^4$. For high-aspect ratio beams such as those found in LIGA or DRIE silicon, b/a is often 0.1 or less, in which case an approximation of

$$K \approx 5ab^3 \tag{EQ 44}$$

is accurate to within a five percent for any beams routinely encountered.

The maximum shear stress occurs at the midpoints of the two longer sides, and is given by $\tau_{max} \approx \frac{3M}{8ab^2}$ for high aspect ratio beams, $\tau_{max} = \frac{0.6M}{a^3}$ for square beams, and $\tau_{max} = \frac{2M}{\pi r^3}$ for a circle.

Note that since the maximum shear strain is a property of the cross-section only, and the spring constant is a function of the length, that the angular deflection at which the beam fractures in pure torsion is proportional to length. For a given cross-section we can look at the maximum deflection in radians (of torsional deflection) per meter (of length)

$$\theta_{fracture} = \frac{M_{fracture}}{K_\theta} = \frac{\frac{8ab^2}{3}\tau_{max}}{\frac{5ab^3G}{L}} = \frac{8}{15}\frac{L}{b}\frac{\tau_{max}}{G} \tag{EQ 45}$$

where again 2b is the short dimension of the rectangular cross section.

Example: Calculate the torsional stiffness of an isotropic beam with a shear modulus of 80GPa, a length of 500 microns, and a cross section of 2 by 40 microns. Also calculate the deflection and applied moment corresponding to failure if the shear strain limit is 800MPa.

Solution: From Eqn. 42 and Eqn. 44 we find that

$$K_\theta = \frac{KG}{L} \approx \frac{5ab^3G}{L} = \frac{5(2\times10^{-5}m)(1\times10^{-6}m)^3\left(8\times10^{10}\frac{N}{m^2}\right)}{5\times10^{-4}m}$$

$$= \frac{5(2)(8)}{5}10^{-5-18+10+4}\frac{m^4\frac{N}{m^2}}{m} = 16\times10^{-9}Nm$$

The maximum shear strain occurs on the midline of each of the 40 micron sides, and has a magnitude of

$$\tau_{max} = \frac{3M}{8ab^2} = \frac{3}{8(2\times10^{-5}m)(1\times10^{-6}m)^2}M = \frac{3}{8(2)m^3}10^{5+12}M \approx 1.9\times10^{16}\frac{1}{m^3}M$$

so the torque corresponding to the strain limit is given by

$$M_{fracture} = \frac{\tau_{max}}{1.9\times10^{16}\frac{1}{m^3}} = \frac{8\times10^8\frac{N}{m^2}}{1.9\times10^{16}\frac{1}{m^3}} \approx 2\times10^{-8}Nm$$

which corrseponds to a deflection

$$\theta_{fracture} = \frac{M_{fracture}}{K_\theta} = \frac{2\times10^{-8}Nm}{16\times10^{-9}Nm} = 1.25rad$$

(EQ 46)

References

Young, W., Roark’s Formulas for Stress and Strain, 6th edition, McGraw-Hill, 1989.

Problems

1. What is the ratio of the transverse stiffness of a square beam to its axial torsional stiffness? To its transverse torsional stiffness?
2. A pure force is to be applied at some point of the cross-section at the end of a cantilevered beam. How should it be applied (location and direction) to maximize deflection? Minimize deflection? To maximize angular deflection? Minimize angular deflection?
3. Same as above, but with a pure moment.