A beam gap actuator exercise

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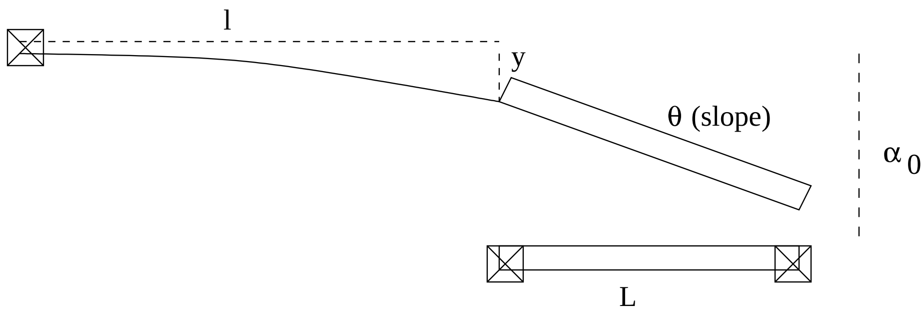
Consider the simple electrostatic actuator represented below. The actuator consists of a thin support beam of length l and width w and two actuator plates of length L and width sufficient to make them appear rigid. The gap between the actuator plates is initially α_0; as voltage is applied, the beam is pulled down by a small amount y and tilted to some small slope θ. The whole assembly is fabricated in a layer of thickness t.



If fringing fields are neglected, the electrostatic attraction between plates in a parallel plate capacitor is given by

$$F = \frac{1}{2}\epsilon_0 A V^2 g^{-2}$$

where A is the area of the two plates, V is the applied voltage, g is the distance across the gap, and ϵ_0 is the permittivity of free space (a different constant would be used in a different medium).

When the plates are at a small tilt, a reasonable approximation of the attraction is given by breaking the plates into little differential elements and treating each element as though it consisted of parallel plates. (In fact, derivation of the potential using this approximation is given in chapter 30 of Halliday and Resnick's general physics text. A similar approximation was used by Hornbeck in analyzing the design of the DMD micromirror; see Senturia's book.)

Using this approximation, we get a distributed load $F(x)dx$ along the length L of the beam:

$$F(x)\,dx = \frac{1}{2}\epsilon_0 t V^2 g(x)^{-2}$$

If g is a constant function, integrating this formula from 0 to L gives the formula for parallel plate attraction from the last paragraph.

For convenience, assume that the two plates are rigid, so that $g(x) = \alpha + \theta x$. We will denote the constant term in the electrostatic attraction by

$$K_e = \frac{1}{2}\epsilon_0 t V^2$$

Then the total force on the top plate is given by

$$F = -K_e \int_0^L (\alpha + \theta x)^{-2}\, dx$$

We can nondimensionalize the integrand by the change of variables $\hat{x} = x/L$ to get

$$\begin{aligned} F &= -K_e L^{-1} \int_0^1 (\alpha/L + \theta \hat{x})^{-2}\, d\hat{x} \\ &= -K_e L^{-1} \hat{F} \end{aligned}$$

Similarly, we can write the moment experienced at the left end of the top plate as

$$\begin{aligned} M &= -K_e \int_0^1 \hat{x}(\alpha/L + \theta \hat{x})^{-2}\, d\hat{x} \\ &= -K_e L^{-1} \hat{M} \end{aligned}$$

Now consider what happens when the top plate is supported by a single rigid beam (see figure). Based on the small-deflection elastic beam equation

$$EIy'' = M(x)$$

the deflection and slope of the right end of the support beam under a point force F and moment M should be

$$\begin{aligned} y' = \theta &= \frac{Ml}{EI} + \frac{Fl^2}{2EI} \\ y &= \frac{Ml^2}{2EI} + \frac{Fl^3}{3EI} \end{aligned}$$

Plugging in the expressions derived above for F and M and doing some algebra gives us

$$\begin{aligned} \theta &= \frac{K_e l}{EI}\left(\hat{M} + \frac{1}{2}\hat{F}\frac{l}{L}\right) \\ y &= \frac{K_e l^2}{EI}\left(\frac{1}{2}\hat{M} + \frac{1}{3}\hat{F}\frac{l}{L}\right) \end{aligned}$$

The first equation is completely non-dimensionalized; we can non-dimensionalize the second by defining $\hat{y} = y/l = y/(Ls)$ where $s = l/L$. We also define the non-dimensional constant $K_n = -K_e l/(EI)$. Finally, we write the non-dimensionalized gap distance function

$$\hat{g}(x) = \hat{\alpha}_0 + s\hat{y} + \theta\hat{x}$$

where $\hat{\alpha}_0$ is the (scaled) initial gap before any actuation. The final equations describing the equilibrium position of the beam, then, are

$$\begin{bmatrix} \theta \\ \hat{y} \end{bmatrix} = K_n \int_0^1 \hat{g}(\hat{x})^{-2} \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix} \begin{bmatrix} \hat{x} \\ s \end{bmatrix} d\hat{x}$$

The behavior of our model to a particular applied voltage should depend only on s and $\hat{\alpha}_0$ (which measure relative dimensions in the device) and K_n. We can write out K_n as

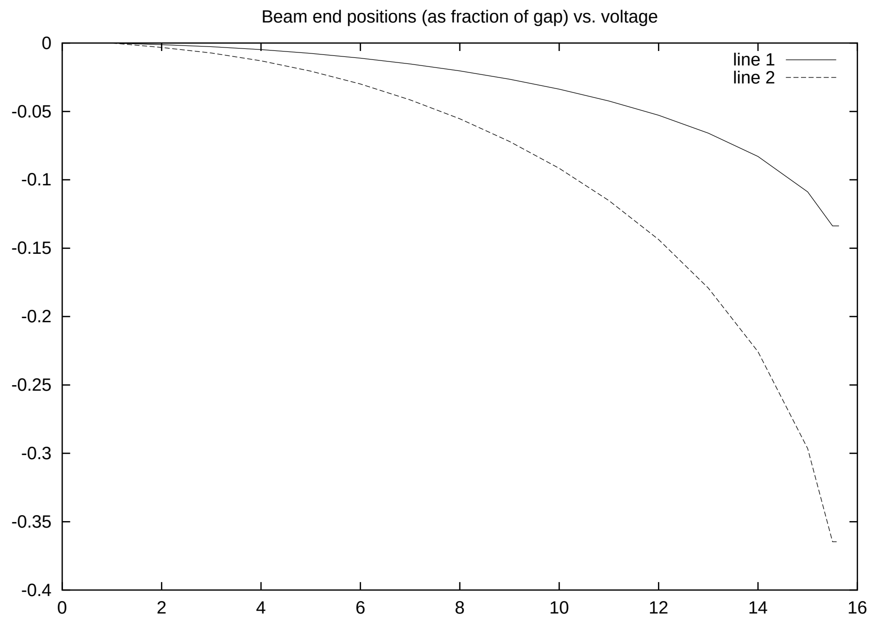
$$K_n = \frac{K_e l}{EI} = \frac{(1/2)\epsilon_0 t V^2 l}{Etw^3/12} = 6\frac{\epsilon_0}{E}\frac{l}{w^3}V^2$$

so the behavior of the model should depend only on

- K_n, which is a simple function of the physical parameters ϵ_0 and E, the applied voltage, and the length and width of the support beam.
- The ratio of the support beam length to the plate length s
- The ratio of the gap size to the plate length $\hat{\alpha}_0$

This observation provides a possible test for correct implementation of the analysis routines and of the model. If we take one beam-gap system and scale the beam lengths and gap size by a factor of 2 and the support beam width by a factor of $2^{1/3}$, we should get the same relative deflection and the same pull-in point. If the behavior is different, there is probably a bug.

It might also be interesting to visualize the implications of the scaling in this simplified model. Consider, for example, the question of pull-in voltage. It is obvious from the above equations that the model pull-in value for K_n is a function of the the ratios $\hat{\alpha}_0$ and s. Because the number of parameters is low, this function is simple to visualize; it's just a surface or contour plot. I have not yet coded this.



To sanity-check my calculations, I wrote a simple Octave program to solve for displacement versus voltage in a particular device. The support beam and plate were both 100 μm long, the support beam width was 2 μm, and the initial gap was 2 μm. I assumed a Young's modulus of 165 GPa. The integrals were computed using an adaptive quadrature subroutine from QUADPACK, and the nonlinear equations were solved using Octave's fsolve, which is based on the MINPACK subroutine hybrd. The voltage increased by one volt per step (when possible), and used the previous step's solution as a guess for the solution at the next step. When fsolve failed to converge, the size of the voltage step was cut by 0.5 and the procedure repeated.

Shown above is the resulting plot of displacement of the beam ends with respect to voltage. The procedure stopped when the step size was reduced to 2^{-10}, which occurred at roughly 15.63 V. This is a substantially higher "numerical pull-in" than was experienced with the pull-in function in SUGAR 1.1; it is not clear whether the difference is due to incorrect implementation in the current experiment, incorrect implementation in the SUGAR 1.1 code, or some combination. However, the general behavior (deflection to roughly 1/3 the initial gap, followed by pull-in) mimics our intuition from the spring-gap parallel plate model.