SUGAR: Three Dimensional Plate element

Tarig I Dinar,UC Berkeley Extension
tarigdinar@yahoo.com

Three dimensional plate element with four nodes was introduced to SUGAR library. The element is based on Bogner-Fox method for generating inter-element compatible stiffness and mass matrix by the use of interpolation formulas. The element was verified for Static, Modal analysis, the damping matrix has been developed but not verified.

Introduction:

Plates are vastly used in MEMS structures as resonating elements, proof masses, and flexures. Plates can be considered as an array of parallel beams fused together, and their interaction opposes dimensional changes, resulting in greater resistance to bending. The Poisson ratio is important for plates as the lateral strain serves to stiffen them, Thus plates will have less curvature than a beam under equilibrium load, approximately $(1-\nu^2)$ as much as plate bending, which can constitute a difference in results while modeling systems in SUGAR since plates are modeled using a two nodes beam element.

Bogner-Fox plate model uses the concept of minimum potential energy in the analysis of plates and shells with a finite number of degrees of freedom, the model insures completely compatible displacement states and geometrical admissibility between the elements.

Plate theory:

The major assumptions made in the derivation of this plate element is **[2]**:

- Normals to the middle surface of the plate remain normal through the deformation.
- Transverse shear deformation is neglected.
- The normal displacement is a function only of the middle surface coordinates.

The above assumptions permit displacement of the order of several times the thickness, but restrict the class of the two dimensional structures to thin plates.

The strain energy for this plate element can be decoupled into membrane and bending, which are represented by the following equations:

$$U_m = \frac{K}{2}\iint_A [u_x^2 + v_y^2 + 2\nu(u_x v_y) + \frac{1}{2}(1-\nu)(u_y + v_x)^2]\,dxdy$$

$$U_b = \frac{K}{2}\iint_A [w_{xx}^2 + w_{yy}^2 + 2\nu(u_{xx} v_{yy}) + \frac{1}{2}(1-\nu)\, w_{xy}^2]\,dxdy$$

where

$$K = \frac{Eh}{1-\nu^2} \quad D = \frac{Eh^3}{12(1-\nu^2)}$$

u, v, and w are the displacements in the x, y and z directions, These displacement modes can be described by the following:

$$u(x,y) \approx \tilde{u}(x,y) = \sum_{i=1}^{2}\sum_{j=1}^{2} \overset{(o)}{H}_{oi}(x)\ \overset{(o)}{H}_{oi}\ b_{ij}$$

v(x,y) will have a similar equation.

where

$$\overset{(o)}{H}_{o1}(s) = -\frac{1}{a}(s-a),\ \overset{(o)}{H}_{o2}(s) = \frac{1}{a}s$$

$$w(x,y) \approx \tilde{u}(x,y) = \overset{(1)}{H}_{oi}\overset{(1)}{H}_{oj} w_{ij} + \overset{(1)}{H}_{oi} w_{1i} \overset{(1)}{H}_{0j} w_{xij}$$
$$\overset{(1)}{H}_{oi} w_{ij} \overset{(1)}{H}_{1j} w_{yij} + \overset{(1)}{H}_{1i}\overset{(1)}{H}_{1j} w_{xyij}$$

where

$$\overset{(1)}{H}_{o1}(s) = \frac{1}{a^3}(2s^3 - 3as + a^3),\ \overset{(1)}{H}_{o2}(s) = -\frac{1}{a^3}(2s^3 - 3as^2)$$

$$\overset{(1)}{H}_{11}(s) = \frac{1}{a^2}(s^3 - 2as^2 + a^2 s),\ \overset{(1)}{H}_{12}(s) = \frac{1}{a^2}(s^3 - as^2)$$

$\overset{(o)}{H}, \overset{(1)}{H}$ are Lagrange and the oscillatory interpolation functions.

Static Analysis:

The membrane and bending stiffness matrices are obtained by constructing the total bending and potential energy for the element and taking the partial derivatives with respect to the independent degrees of freedom and setting these equal to zero. In the case of the membrane stiffness matrix, the independent degrees of freedom are u_i, v_i , while in the case of the bending stiffness matrix, the independent degrees of freedom are $w, w_x, w_y \, and \, w_{xy}$ which are the lateral displacement, rotation around the y-axis, rotation around the x-axis, and the twist rotation.

The displacement vector of the plate at each node is given as

$$\begin{bmatrix} u_{ij} \\ v_{ij} \\ w_{ij} \\ w_{xij} \\ w_{yij} \\ w_{xyij} \end{bmatrix} = \begin{bmatrix} dx \\ xy \\ dz \\ dy \\ dx \\ dz \end{bmatrix}$$

The minimum potential method produces a 24 by 24 stiffness matrix, a MATLAB file was written to generate the matrix using Matlab/symbolic, which is provided in the appendix.

For more accurate element a higher order interpolation functions can be used in the displacement modes to generate the stiffness matrix.

The structure used to demonstrate the static solution is a clamped plate subject to a uniform load of 0.2 Psi for several ratios of length to width. The results analytical solution obtained from **[*]** in the table.1. The analytical solution for the displacement at the center of a 20x20x0.1 in^3 plate that has E=10.92Psi and v=0.3 is 0.402in, the plate element shows less error by using more elements-(shown in table.2).

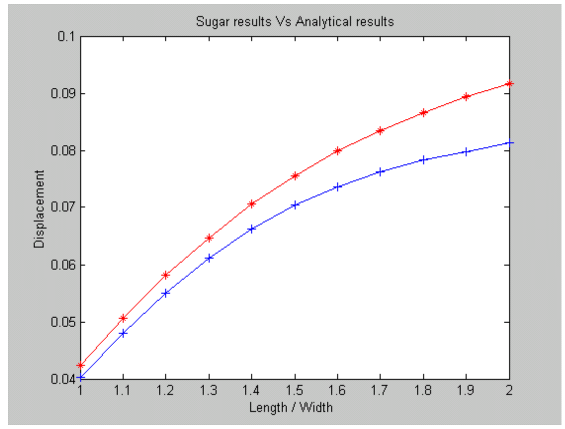

Fig.1: comparison of analytical and SUGAR results for various width /length ratios.

elements	Analytical	Error%
4	0.04329	5.4
16	0.0417	3.7
36	0.403	0.7

Modal analysis

The energy associated with the transverse displacement can be evaluated in the absence of external forces using D'Alembert formulae:

$$W_b = \iint_A -\rho_m h \, \ddot{w} \, w \, dx \, dy$$
$$= \iint_A \rho_m h \, \omega^2 \, w^2 \, dx \, dy$$

Where ρ_m is the mass density of the plate material. When the assumed modes are substituted into the potential energy and the stationary conditions taken, the mass matrix is obtained as:

$$Q_b W = \omega^2 M_b W$$

Where M_b is the consistent mass matrix for the element.

The stiffness and the consistent mass matrices are given in **[2]**.

The plate element results were verified using a plate with one clamped edge and all the other edges are free. The structure was modeled with 16 plate elements.The plate dimensions are 10x10x0.01, with a density of 1 and 10.92 E6Psi elastic modulus **[3]**.

Frequency	Analytical	Sugar
ω_1	0.3471	0.3475
ω_2	0.8508	0.8517
ω_3	2.128	2.1325
ω_4	2.719	2.725
ω_5	3.095	3.104
ω_6	5.418	5.446

Table.3: comparison of analytical and SUGAR results for the six first modes.

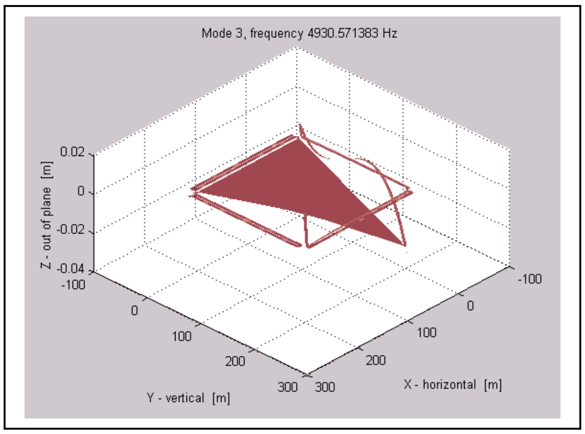

Fig.2 Mode 3 of a modal analysis of parallel plate accelerometer rotator using the plate and beam elements, the figure displays the plate using the display functions developed for plates.

Transient analysis:

The system of equations governing the transient analysis is of the form:

$$[M]\{\ddot{q}\}+[D]\{\dot{q}\}+K\{q\}=\{F(t,\{q\})\}$$

Where

$K \equiv$ Stiffness matrix.

$C \equiv$ Damping matrix

$M \equiv$ Mass matrix

Viscous air damping is the dominant dissipation mechanism for microstructures that operate at atmospheric pressure. The damping matrix can be decoupled into membrane and bending damping matrices.

Squeeze film damping from vertical motion creates a pressure in the thin film of air between the plate and the substrate. Isothermal, small pressure-variation and small displacements assumptions applied to the fluid governing equation result in the expression **[5]**:

$$\frac{d^2P}{dx^2}+\frac{d^2P}{dy^2}=\frac{12\mu}{w_o}\frac{d(w)}{dt}$$

Where

$P \equiv$ Squeeze film pressure.

$\mu \equiv$ Air viscosity

$w_o \equiv$ Air gap height

$d(w) \equiv$ Plate displacement.

The squeeze film-damping coefficient for the rectangular plate is **[6]**:

$$\overline{C}_b=\kappa(W/L)\frac{\mu a b^3}{w_o^3}$$

Where $\kappa(W/L)$ the effect of the plate width/length ratio in damping.

Curve fitting was used to obtain the function that describes the effect of plate dimensions on squeeze film damping shown in fig.3 **[5]**

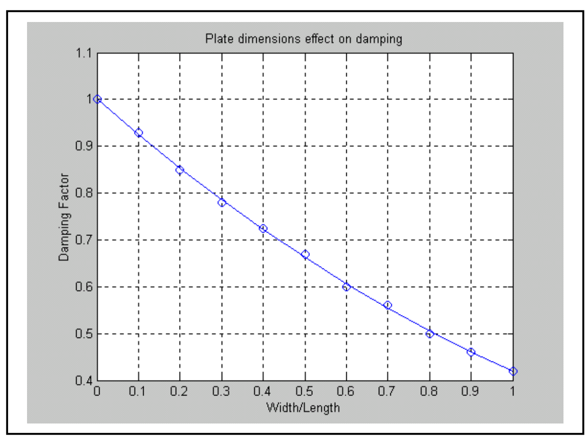

Fig.3 Effect of width/Length ratio on Damping produced by a rectangular plate.

The potential energy due to damping is given by:

$$C_b = \frac{\overline{C}}{2} \iint_A (\dot{W})^2 \, dxdy.$$

From the similarity with the consistent matrix derivation, the bending damping matrix can be presented in:

$$D_b = \overline{C}_b \frac{M_b}{\rho h}.$$

Where M_b is the consistent bending mass matrix, D_b is the Damping bending matrix.

For deriving the membrane-damping matrix, a Coette gas flow damping effect was used as in:

$$\overline{C}_m = \frac{\mu a b}{w_o}.$$

again the membrane damping matrix will be given by:

$$D_m = \overline{C}_m \frac{M_m}{\rho h}.$$

Where M_m is the consistent membrane mass matrix for the element, D_m is the membrane Damping matrix.

Conclusions:

A four nodded plate element was introduced to SUGAR. The element was verified for the static and modal analysis. The damping matrix was developed for low order damping but not verified. A display function was also introduced for this plate element.

Acknowledgment

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