Today’s Lecture

• The cantilever beam under small deflections
• Combining cantilevers in series and parallel: folded suspensions
• More accurate models: large deflections, shear, …
• Design implications of residual stress and stress gradients

• Reading:
Macro and Milli Suspensions

2000 Ford Focus
mostly 3-D steel parts and assembly-line production
... 100,000's per year

Hard Disk Suspensions
(stamped 20 μm stainless steel with laminated 10 μm polyimide + 15 μm copper interconnect)
... 1,000,000's per week

Springs in MEMS

- Coils: 3-D is tough for planar processing!

- Flexures: straightforward to make using surface or bulk micromachining, but details of fabrication process constrain dimensions and anchors/joints

- Simplest flexure: a “clamped-free” cantilever beam ... a.k.a. a diving board
A Cantilever Beam

Clamped: $y = 0, \frac{dy}{dx} = 0$ at $x = 0$

$x = L_c$

$F$

**Goal:** find relation between tip deflection $y(x = L_c)$ and applied load $F$

**Assumptions:**

1. Tip deflection is small compared with beam length
2. Plane sections (normal to beam’s axis) remain plane and normal during bending … “pure bending”
3. Shear stresses are negligible

Checking the Assumptions

J.-A. Schweitz, Uppsala University
A Beam Segment in Pure Bending

- Top is in tension
- Bottom is in compression
- Neutral axis ($\varepsilon_x = 0$)

Bending Moment $M_z$

- Concept of moment (basic physics): $\text{force} \times \text{distance}$
- Integrate stress through thickness of beam

$$-M_z = \int_{-h/2}^{h/2} [(W dy)\sigma_x] \cdot y = EW \int_{-h/2}^{h/2} \varepsilon_x y dy$$

Why a minus sign? See Senturia, pp. 208-210

$$\varepsilon_x = \varepsilon_{x,\text{max}} \left( \frac{y}{h/2} \right)$$
Bending Strain and Beam Curvature

\[-M_z = EW \int_{-h/2}^{h/2} \varepsilon_{x,max} \left( \frac{y}{h/2} \right) dy = \left( EW \frac{h}{2} \right)^3 \varepsilon_{x,max} \]

Radius of curvature \( \rightarrow \) geometric connection to strain

\[\varepsilon_{x,max} = \frac{(R + h/2)d\theta - Rd\theta}{Rd\theta} = \frac{h/2}{R} \]

Curvature and Strain (cont.)

\[R^{-1} = \frac{d^2 y}{dx^2} \quad \text{… result from basic calculus} \]

Combining the curvature and moment results:

\[\varepsilon_{x,max} = \frac{h/2}{R} \quad \text{and} \quad -M_z = \left( EW \frac{h}{2} \right)^3 \varepsilon_{x,max} \]

\[-M_z = \left( EW \frac{h}{2} \right)^3 \left( \frac{h/2}{R} \right) = \frac{1}{12} EWh^3 R^{-1} = \frac{EWh^3}{12} \frac{d^2 y}{dx^2} \]
Flexural Rigidity (Moment of Inertia) $I_z$

- The term $Wh^3/12$ is defined as the flexural rigidity, $I_z$ (Senturia uses “moment of inertia”)

- Large flexural rigidity $\rightarrow$ low curvature $\rightarrow$ small deflections $\rightarrow$ stiff

$$-M_z = EI_z \frac{d^2y}{dx^2}$$

- Design implications:
  1. rigidity increases as the cube of the beam’s thickness (in the direction of bending)
  2. the aspect ratio $h/W$ determines the ratio of bending rigidity in the $y$ and the $z$ directions

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Revisit Cantilever Deflection due to Residual Stress Gradients

- Model the strain by a linear profile $\varepsilon_{res}(y) = \bar{\varepsilon}_{res} + \Gamma y$

LPCVD poly-Si: measured* stress profiles


* inferred from wafer curvature after incremental thinning of poly-Si
Built-in Bending Moment

- Integrate differential moment through film thickness (sign?)

\[ M_r = \int_{-h/2}^{h/2} \sigma_r (Wdy) y = EW \int_{-h/2}^{h/2} \varepsilon_r (y) \cdot y dy \]

\[ M_r = EW \int_{-h/2}^{h/2} (\varepsilon_r + \Gamma y) \cdot y dy = 0 + \left( \frac{EWh^3}{12} \right) \Gamma \]

- Apply moment to the cantilever \( \rightarrow \) constant curvature

\[ \left( \frac{EWh^3}{12} \right) \Gamma = EI \frac{d^2 y}{dx^2} \]

\[ \Gamma = \frac{d^2 y}{dx^2} \]

Tip-Deflection (Small Deflections)

- Integrate to find the tip deflection \( y(x = L) \)

\[ y(x = L) = \Delta = \frac{1}{2} \Gamma L^2 \]

- The strain gradient \( \Gamma \) can be found from the tip deflection \( \Delta \):

\[ \Gamma = 2 \frac{\Delta}{L^2} \]
Boundary Conditions

- A “step-up” anchor will result in the average strain causing an offset angle at \( y = 0 \)

The Cantilever with a Concentrated Load

Clamped: \( y = 0, \ dy/dx = 0 \) at \( x = 0 \)

Find the tip deflection \( y(x = L_c) \) and applied load \( F \) ... get effective spring constant \( k_c \)

\[-M_z(x) = EI_z \frac{d^2 y}{dx^2}\]

The moment varies linearly with \( x \)

\[-M_z(x) = F(L_c - x)\]
Tip Deflection

- Integrate ODE twice and apply boundary conditions (zero displacement, zero slope) at anchor

\[ y(x) = \left( \frac{F}{6EI} \right)(3L_c - x)x^2 \]

- Tip deflection: \( y(L_c) \)

\[ y(L_c) = \left( \frac{F}{3EI} \right) L_c^3 \]

- Spring constant: \( k_c \) (N/m) \( \ldots = (\mu N/ \mu m) \)

\[ k_c = \left( \frac{3EI}{L_c^3} \right) = \frac{EWh^3}{4L_c^3} \]

Summary of Common Loading and Boundary Conditions

Compendium of useful results:
http://www.roarksformulas.com

Series Combinations of Cantilevers

- Springs in series $\rightarrow$ same load; deflections add

$$y(L) = \frac{F}{k} = 2 \quad y(L_c) = 2 \left( \frac{F}{k_c} \right) = F \left( \frac{1}{k_c} + \frac{1}{k_c} \right)$$

Compliances add

$$\frac{1}{k} = \frac{1}{k_c} + \frac{1}{k_c}$$

Parallel Combinations of Springs

- Same displacement $\rightarrow$ load is shared and the spring constant is the **sum** of the individual spring constants

$$y(L) = \frac{F}{k} = \frac{F_a}{k_a} = \frac{F_b}{k_b} = \left( \frac{F}{2} \right) \left( \frac{1}{k_a} \right)$$

$$k = 2k_a$$
Folded-Flexure Suspension Variants


Folded Flexure Deflection

Overall Spring Constant

- Four pairs of clamped-guided beams, each of which bend in series (assume that trusses are inflexible)

Force is shared by each pair $\Rightarrow F_{pair} = F/4$

Displacement of two legs add (springs in series) $\Rightarrow$

$y = F_{pair}/k_{pair} = (F/4)(1/k_{leg} + 1/k_{leg})$

$1/k_{leg} = 1/k_c + 1/k_c = 2/k_c$

$y = (F/4)(2/k_c + 2/k_c) = F/k_c$

Selected Goals for Suspension Design

- Compliance ratios are often required to be large (e.g., the comb drive’s maximum force is determined by lateral instability, which is in turn directly related to the lateral spring constant)

- Undesirable resonant modes of the structure are often required to be at significantly higher frequencies, which translates to stiffer spring constants

- Robustness against residual stress and stress gradients (e.g., folded flexures release most of the residual stress and cancel deflections due to gradients)
Folded Flexure Suspension with Residual Stress Gradient

Michael Judy, Ph.D. Thesis
EECS Dept., UC Berkeley, 1994

ADXL-50 Suspension

- Straight tethers are pulled flat under tension, “backbone” domes up about 0.5 μm; sense fingers are curled downward off of backbone
ADXL-05 Suspension

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BiMEMS Foundry Process

- Analog Devices, 1994 – circa 2000
- $\sigma_r \approx 50 \text{ MPa} - [5 \text{ MPa/}\mu\text{m}](z) \rightarrow$ cantilevers bend down

Z-Axis Accelerometer Warpage

- Anchor placement can be critical for balancing doming of proof mass and warpage of folded suspension
Motorola z-Axis Accelerometer

Limits to Linearity

- Cantilever beams: stiffen as the deflection exceeds about 10% of the \textit{length} of the beam
- Note: clamped-clamped beams deviate from non-linearity for much smaller deflections (next lecture)

Michael Judy, Ph.D. Thesis
EECS Dept., UC Berkeley, 1994
When is Shear Significant?

- Fixed-guided beam under point load (two cantilevers in series)

\[ k_x^{-1} = k_{xb}^{-1} + k_{xs}^{-1} \]

\[ k_{xb} = \frac{12E I_z}{L^3} \quad k_{xs} = \frac{GWh}{1.5 L} = \frac{EWh}{1.5(1 + \nu)L} \]

Solving for \( k_x \):

\[ k_x = \frac{k_{xb}}{1 + 3.8 \left( \frac{W}{L} \right)^2} \quad (L > 10W \rightarrow 2.5\% \text{ or less error}) \]


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