Today’s Lecture

• Exact analysis of suspensions is too messy for design purposes: an approximate method that gives insight into 2nd order effects would be very useful

• Energy method (principle of virtual work) is a powerful approach for gaining insight for suspension design

• Examples: tapered cantilever beam, amplitude-stiffened clamped-clamped beam

• Reading:
Clamped-Guided Beam Under Axial Load

- Important case for MEMS suspensions, since the thin films from which they’re made are often under residual stress
- Consider small-deflection case: \( y(x) \ll h \)

Differential equation:

\[
EI_z \frac{d^4 y}{dx^4} - S \frac{d^2 y}{dx^2} = F \delta(x - L)
\]

Solving the ODE


- \( S > 0 \) (tension)
  \[
  k^{-1} = \frac{pL - 2 \tanh(pL/2)}{p|S|} = \frac{y(x = L)}{F}
  \]

- \( S < 0 \) (compression)
  \[
  k^{-1} = \frac{-pL + 2 \tan(pL/2)}{p|S|} = \frac{y(x = L)}{F}
  \]

where \( p = \sqrt{\frac{|S|}{EI_z}} \)
### Design Implications

- **Straight flexures** →
  
  Large tensile $S$ means flexure behaves like a tensioned wire ($k^{-1} = L/S$); large compressive $S$ can lead to buckling ($k^{-1} \to \infty$)

- **Folded flexures** →
  
  Residual stress is only partially released since length from truss to shuttle’s centerline is different by $L_s$ for inner and outer legs

### Effect on Spring Constant

- Residual compression on outer legs with same magnitude of tension on inner legs →
  
  $\varepsilon_b = \pm \varepsilon_r (L_s/L)$ and $S = \pm E\varepsilon_r (L_s/L) \text{ Wh}$

  $$k = 4(k_{com}^{-1} + k_{ten}^{-1})^{-1}$$

  $$k = 4 \left[ \frac{-pL + 2\tan(pL/2)}{p|S|} + \frac{pL - 2\tanh(pL/2)}{p|S|} \right]^{-1}$$

- Reduce the shoulder width $L_s$ to minimize stress in legs …limits before rigidity of shuttle is compromised

- Compliance in the truss lowers the axial compression and tension and reduces its effect on the spring constant
Principle of Virtual Work

- In an energy-conserving system (i.e., elastic materials), then the energy stored in a body due to the quasi-static (i.e., slow) action of surface and body forces is equal to the work done by these forces …
- Implication: we can vary the surface and body forces to suit our convenience and find the minimum of the difference $U$ between the stored energy and the work done by the forces:

$$U = \text{Stored Energy} - \text{Work Done}$$

**Key idea:** we don’t have to reach $U = 0$ to produce a very useful, approximate analytical result for load-deflection

Energy Density

Strain energy density (J/m$^3$): find work done in straining material

$$w = \int_0^{\varepsilon_x} \sigma_x \, d\varepsilon_x \quad \text{x-axis normal stress term}$$

$$w = \int_0^{\varepsilon_x} E \varepsilon_x \, d\varepsilon_x = \frac{1}{2} E \varepsilon_x^2$$

Total strain energy (J):

$$W = \iiint \left( \frac{1}{2} E \left( \varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 \right) + \frac{1}{2} G \left( \gamma_{xy}^2 + \gamma_{xz}^2 + \gamma_{yz}^2 \right) \right) dV$$
Bending Energy Density

\[ y(x) = \text{transverse displacement of neutral axis} \]

Bending energy \( dW_{\text{bend}} \) in length \( dx \):

\[
dW_{\text{bend}} = Wdx \int_{-h/2}^{h/2} \frac{1}{2} E \varepsilon_x^2 (y') dy' \quad \varepsilon_x (y) = y' \frac{d^2 y}{dx^2}
\]

\[
dW_{\text{bend}} = Wdx \int_{-h/2}^{h/2} \frac{1}{2} E \left( y' \frac{d^2 y}{dx^2} \right)^2 dy' = \frac{1}{2} E \left( \frac{Wh^3}{12} \right) \left( \frac{d^2 y}{dx^2} \right)^2 dx
\]

\[
W_{\text{bend}} = \frac{1}{2} EI_z \int_{0}^{L} \left( \frac{d^2 y}{dx^2} \right)^2 dx
\]

Energy due to Axial Load

Strain due to axial load \( S \) contributes an energy \( dW_{\text{stretch}} \) in length \( dx \):

\[
dS = \left[ (dx)^2 + (dy)^2 \right]^{1/2} = dx \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{1/2} \approx dx \left[ 1 + \frac{1}{2} \left( \frac{dy}{dx} \right)^2 \right]
\]

\[
\varepsilon_x = \frac{ds - dx}{dx} = \frac{1}{2} \left( \frac{dy}{dx} \right)^2
\]
Axial Strain Energy (Cont.)

\[ dW_{\text{axial}} = S\varepsilon_x \, dx = \frac{1}{2} S \left( \frac{dy}{dx} \right)^2 dx \to W_{\text{axial}} = \frac{1}{2} S \int_0^L \left( \frac{dy}{dx} \right)^2 dx \]

Shear Strain Energy*

\[ W_{\text{shear}} = \frac{3(EI_x)^2}{4GWh} \int_0^L \left( \frac{d^3 y}{dx^3} \right)^2 dx \]


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Applying the Principle of Virtual Work

- Guess the form of the beam deflection under the applied loads … how??
- Then vary the parameters in the beam deflection function in order to minimize:

\[ U = \sum W_j - \sum F_i u_i \]

Assumes point load

Displacement at point load

See Senturia, p. 244, for general expression with distributed surface loads and body forces
Guidelines for the Deflected Shape $y(x)$

- Satisfies the boundary conditions
- Strain energy integrals aren’t too tedious (math software can modify this point)
- Example: tapered cantilever beam

$$y(x) = c_2 x^2 + c_3 x^3$$

Example: tapered cantilever beam

Top view of cantilever’s $W(x)$

$$x = L_c$$

Strain Energy and Work by $F$

$$W_{bend} = \frac{1}{2} E \int_0^{L_c} I_z(x) \left( \frac{d^2 y}{dx^2} \right)^2 dx$$

$$I_z(x) = \frac{W(x) h^3}{12}$$

$$\frac{d^2 y}{dx^2} = 2c_2 + 6c_3 x$$

$$W(x) = W \left(1 - \frac{x}{2L_c}\right)$$

Example: 50% taper

$$U = W_{bend} - F \cdot y(L_c) = \frac{1}{24} Ewh^3 \int_0^{L_c} \left(1 - \frac{x}{2L_c}\right)(2c_2 + 6c_3 x)^2 dx - F \left(c_2 L_c^2 + c_3 L_c^3\right)$$

Tip deflection
Minimum $U \rightarrow$ Find $c_2$ and $c_3$

- $U$ minimization $\rightarrow$ get the closest approximation to zero (what it would be if we had guessed correctly)
- For each parameter separately, the partial derivative of $U$ with respect to the parameters $c_2$ and $c_3$ must be zero:

$$ \frac{\partial U}{\partial c_2} = 0 \quad \frac{\partial U}{\partial c_3} = 0 $$

After evaluating the integral:

$$ U = EWh^3 \left\{ \frac{5c_3^2}{16} L_c^3 + \frac{c_2c_3}{3} L_c^2 + \frac{c_2^2}{8} L_c \right\} - F\left(c_2^2 L_c^2 + c_3 L_c^3\right) $$

Solve Simultaneous Equations

$$ \begin{align*}
\frac{\partial U}{\partial c_2} &= 0 = \left(\frac{EWh^3}{3} c_3 - F\right) L_c^2 + \left(\frac{EWh^3}{4} c_2\right) L_c \\
\frac{\partial U}{\partial c_3} &= 0 = \left(\frac{5}{8} EWh^3 c_3 - F\right) L_c^3 + \left(\frac{EWh^3}{3} c_2\right) L_c^2
\end{align*} $$

$$ c_2 = \left(\frac{84}{13}\right) \frac{FL_c}{EWh^3} \quad c_3 = -\left(\frac{24}{13}\right) \frac{F}{EWh^3} $$

$$ y(x) = \left(\frac{24F}{13EWh^3}\right) \left(\frac{7}{2}\right) L_c x^2 $$
Spring Constant for Tapered Cantilever

- Solve for tip deflection and define spring constant:

\[ y(L_c) = \left( \frac{24F}{13EWh^3} \right) \left( \frac{5}{2} \right) L_c^3 \]

\[ k_c = F / y(L_c) = \left( \frac{13EWh^3}{60L_c^3} \right) \]

- Constant-width cantilever beam (Lecture 7):

\[ y(L_c) = \left( \frac{F}{4EWh^3} \right) L_c^3 \]

(13% reduction … why so small?)

Ansys Simulation

\[ L = 500 \mu m \]
\[ W = 20 \mu m, \text{ (base; tip = 10 } \mu m) \]
\[ h = 2 \mu m \]
\[ E = 170 \text{ GPa} \]

Static analysis \rightarrow
\[ k = 0.471 \mu N/ \mu m \]

Matches result from energy minimization to 3 significant figures.
Finding a Better Approximation

- Add more terms to polynomial $\rightarrow$ need Mathematica, Maple, or lots of time …
- Add other strain energy terms:
  - Shear (larger for shorter beam)
  - Axial (large deflections)

- What are the payoffs?
  - Analytical expressions
  - Successive refinement
  - Compare importance of terms

Large Deflections

- Springs often stiffen for large deflections:
  \[ k = k_o + k_1y + k_2y^2 \]
- Example: center-loaded clamped-clamped beam
  Senturia, Section 10.4, pp. 249-253

\[ \text{-L/2} \quad F \quad \text{L/2} \]
Principle of Virtual Work

- Guess the deflection (common approach: use the exact solution for small deflections)
  \[ y(x) = \frac{c}{2} [1 + \cos\left(\frac{2\pi x}{L}\right)] \]

- Additional energy for large deflections: “self-generated” axial strain due to stretching of the neutral axis
  \[ \varepsilon_{axial} = \frac{1}{2L} \int_0^L \left( \frac{dy}{dx} \right)^2 dx \]

Potential Energy Function

- Total strain in beam
  \[ \varepsilon_x = \varepsilon_{x,bend} + \varepsilon_{x,axial} = -y' \frac{d^2 y}{dx^2} + \frac{1}{2L} \int_0^L \left( \frac{dy}{dx} \right)^2 dx \]

- Strain energy
  \[ W = \frac{1}{2} EW \int_0^{h/2} \int_{-h/2}^{h/2} \varepsilon_x^2 dy' dx \]

- Potential energy function
  \[ U = W - Fc = \frac{EWh\pi^4 (8h^2 + 3c^2)c^2}{96L^3} - Fc \]
Amplitude-Stiffened Spring

- Find value of \( c \) that minimizes \( U \)

\[
0 = \frac{\partial U}{\partial c} = \left( \frac{\pi^4}{6} \right) \left( \frac{EWh^3}{L^3} \right) c + \left( \frac{\pi^4}{8} \right) \left( \frac{EWh}{L^3} \right) c^3 - F
\]

- Spring rate = \( k(c) = F/c \)

\[
k(c) = k_o + k_2 c^2 = \left( \frac{\pi^4}{6} \right) \left( \frac{EWh^3}{L^3} \right) + \left( \frac{\pi^4}{8} \right) \left( \frac{EWh}{L^3} \right) c^2
\]

Geometric Nonlinearities

- Radius of curvature

\[
\frac{1}{R} = \frac{-d^2y/dx^2}{[1+(dy/dx)^2]^{3/2}}
\]

- Solutions: elliptic integrals

M. W. Judy, Ph.D. Thesis
UC Berkeley, 1994
Folded Suspensions in Large Deflection

- Fit cantilever solution to 3\textsuperscript{rd} order polynomial in deflection

\[ F = \frac{24EIz}{L^3}(x_o + \frac{0.272}{L^2}x_o^3) \]

M. W. Judy, Ph.D. Thesis
UC Berkeley, 1994