Today’s Lecture

- Mechanical structures under driven harmonic motion
  → develop analytical techniques for estimating the resonant frequency
- Lumped mass-spring case: ADXL-50 example
- Rayleigh-Ritz method and examples:
  lateral resonator, double-ended tuning fork
- Viscous damping
- Damped 2nd order mechanical systems:
  transfer functions

- Reading:
Estimating Resonant Frequencies

Simple harmonic motion

\[ x(t) = x_0 \cos(\omega t) \]

Potential energy:
\[ W(t) = \frac{1}{2} k x^2(t) = \frac{1}{2} k x_0^2 \cos^2(\omega t) \]

Kinetic energy:
\[ K(t) = \frac{1}{2} M \dot{x}^2(t) = \frac{1}{2} M x_0^2 \omega^2 \sin^2(\omega t) \]

Energy Conservation

Ignoring dissipation \( \rightarrow W_{max} = K_{max} \)

\[ W_{max} = \frac{1}{2} k x_0^2 = K_{max} = \frac{1}{2} M \omega^2 x_0^2 \]

Resonant frequency:
\[ \omega = \sqrt{\frac{k}{M}} \]

Applications: lumped masses that dominate the suspension’s mass
Example: ADXL-50

Suspension beam: \( L = 260 \, \mu m, \ h = 2.3 \, \mu m, \ W = 2 \, \mu m \)
Average residual stress through thickness: \( \sigma_t = 50 \, MPa \)

Lumped Spring-Mass Approximation

- Mass = \( M = 162 \) nano-grams, 60% is from the capacitive sense fingers
- Suspension: four tensioned beams ... include both bending and stretching terms

Bending compliance \( k_b^{-1} \)

Stretching compliance \( k_{st}^{-1} \)

A.P. Pisano, BSAC Inertial Sensor Short Courses, 1995-1998
ADXL-50 Suspension Model

- Bending contribution:
  \[ k_b^{-1} = (1/k_c + 1/k_e) = 2 \left[ \frac{(L/2)^3}{3E(Wh^3/12)} \right] = \frac{L^3}{EWh^3} = 4.2 \mu m/\mu N \]

- Stretching contribution:
  \[ k_{st}^{-1} = L/S = \frac{L}{\sigma_f Wh} = 1.14 \mu m/\mu N \]

- Total spring constant: add bending to stretching
  \[ k = 4(k_b + k_{st}) = 4(0.24 + 0.88) = 4.5 \mu N/\mu m \]

ADXL-50 Resonant Frequency

Lumped mass-spring approximation:

\[ f = \frac{1}{2\pi} \sqrt{\frac{k}{M}} = \frac{1}{2\pi} \sqrt{\frac{4.48 N/m}{162 \times 10^{-12} kg}} = 26.5 kHz \]

Data sheet: \( f = 24 \) kHz

Note: we have not included the frequency-pulling effect of the DC bias voltage on the sense capacitor plates
Distributed Mechanical Structures

- Vibrating structure’s displacement function can be separated into two parts:

\[ y(x, t) = \hat{y}(x) \cos(\omega t) \]

- We can use the static displacement of the structure as a trial function and find the strain energy \( W_{\text{max}} \) at the point of maximum displacement \( t = 0, \pi/\omega, \ldots \)

\[ v(x, t) = -\omega y(x) \sin(\omega t) = 0 \]

Maximum Kinetic Energy

- At times \( t = \pi/(2\omega), 3\pi/(2\omega), \ldots \), then the displacement of the structure is \( y(x) = 0 \).

- The velocity of the structure is maximum and all its energy is kinetic (since \( W = 0 \))
Finding the Maximum Kinetic Energy

- A differential length $dx$ along the beam has a kinetic energy $dK$ due to its transverse motion:

$$v(x, t') = -\omega y(x)$$

$$dK = (1/2)(dm)v^2(x, t')$$

$$dm = \rho(Whdx)$$

- Total maximum kinetic energy:

$$K_{\text{max}} = \int_0^L \frac{1}{2} \rho Whdxv^2(x, t') = \int_0^L \frac{1}{2} \rho Wh\omega^2 y^2(x)dx$$

The Raleigh-Ritz Method

- The maximum potential and maximum kinetic energies must be equal $\rightarrow$

$$K_{\text{max}} = \int_0^L \frac{1}{2} \rho Wh\omega^2 y^2(x)dx = W_{\text{max}}$$

- The resonant frequency of the beam is therefore:

$$\omega = \sqrt{\frac{W_{\text{max}}}{\int_0^L \frac{1}{2} \rho Wh\omega^2 y^2(x)dx}}$$
Example: Folded-Flexure Resonator

Lumped masses (shuttle, truss) need to be included in the kinetic energy expression.

Kinetic Energy

- Shuttle mass = $M_s$, truss mass = $M_t$

$$K_{\text{max}} = \frac{1}{2} M_s v_s^2 + \frac{1}{2} M_t v_t^2 + \int_0^L \frac{1}{2} \rho W h \omega^2 \hat{y}^2(x) dx$$

- Use static deflection as estimate of mode shape

$$\hat{y}(x) \equiv y(x) = Y_0 \left[ 3 \left( \frac{x}{L} \right)^2 - 2 \left( \frac{x}{L} \right)^3 \right]$$

- Magnitude of shuttle velocity is $v_s = \omega Y_0$

- Magnitude of truss velocity is $v_t = \omega Y_0/2$
Resonant Frequency

• Find maximum potential energy from spring constant and shuttle deflection:

\[ W_{\text{max}} = \frac{1}{2} k_y Y_o^2 \]

\[ k_y = \frac{24EI_z}{L^3} = \frac{2EWh^3}{L^3} \]

• Rayleigh-Ritz equation:

\[ \omega = \sqrt{\frac{k_y}{M_s + (M_t/4) + (12M_b/35)}} \]

both trusses all 8 beams

Double-Ended Tuning Forks

Trey Roessig, Ph.D., ME Dept., UC Berkeley, 1998
Mode Shapes for Clamped-Clamped Beams

\[ \phi(x) = \kappa [\sinh(\beta x - \sin(\beta x)) + \alpha(\cosh(\beta x) - \cos(\beta x))] \]

### Table

<table>
<thead>
<tr>
<th>Mode</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( \kappa )</th>
<th>( \varepsilon = x/L )</th>
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<tbody>
<tr>
<td>First</td>
<td>-1.018</td>
<td>4.730</td>
<td>-0.618</td>
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<tr>
<td>Second</td>
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<tr>
<td>Third</td>
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<td>-10.996</td>
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</table>

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Applying Rayleigh-Ritz to DETFs

Case 1: \( \sigma_r = 0 \)

Case 2: \( \sigma_r = 30 \text{ MPa} \)

- model comb drive as a point mass
- with \( m = 1 \text{ pgm} \)

Dimensions: \( L = 200 \mu \text{m}, h = W = 2 \mu \text{m}, E = 150 \text{ GPa}, \rho = 2.3 \text{ gmcm}^{-3} \)

Use first mode shape for both cases:
- analyze a single tine

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DETF Results

Bending: \( \int_{0}^{1} \left( \frac{d^2 \phi_1}{d \varepsilon^2} \right)^2 d \varepsilon = 198.6 \)

Axial load: \( \int_{0}^{1} \left( \frac{d \phi_1}{d \varepsilon} \right)^2 d \varepsilon = 4.85 \)

Kinetic energy (distributed along beam): \( \int_{0}^{1} \phi_1^2(\varepsilon) d \varepsilon = 0.397 \)

\[ \omega_1^2 = \frac{EI_z}{L^3} \int_{0}^{1} \left( \frac{d^2 \phi_1}{d \varepsilon^2} \right)^2 d \varepsilon + \frac{\sigma_r \rho L}{Wh} \int_{0}^{1} \left( \frac{d \phi_1}{d \varepsilon} \right)^2 d \varepsilon = \frac{k_{\text{eff}}}{M_{\text{eff}}} \]

Case 1: 412 kHz
Case 2: 339 kHz

Resonant Sensors

The double-ended tuning fork is an excellent sensor for axial stress, since it converts it into a shift in frequency (or in oscillation period) \( \rightarrow \) these are easily and accurately measured.

Therefore, we can use the DETF as a building block for a variety of resonant sensors, such as accelerometers (T. A. Roessig, Ph.D. ME, 1998) and gyroscopes (A. A. Seshia, Ph.D. EECS, 2002)

Rayleigh-Ritz Method yields analytical expressions for frequency shift due to axial stress (residual or applied):

\[ \frac{d \omega_1}{d \sigma_r} = \frac{d}{d \sigma_r} \sqrt{\frac{k_{\text{eff}}}{M_{\text{eff}}}} = 4.85 \left( \frac{k_{\text{eff}}}{M_{\text{eff}}} \right)^{-1/2} \]

\[ \frac{dk_{\text{eff}}}{d \sigma_r} = \frac{1}{2} \frac{1}{\omega_1} \frac{Wh}{L} \int_{0}^{1} \left( \frac{d \phi_1}{d \varepsilon} \right)^2 d \varepsilon \]

design insight!
Damping

- Many sources: ignore internal damping and acoustic radiation from the anchors → focus on drag from the surrounding gas

Gap dimension can be on the order of mean-free path → continuum model doesn’t apply
**Couette Damping**

Drag force: \( F_d = -bv_y \)

Damping coefficient: \( b \)

\[
b = \frac{\mu A}{y_o}
\]

Effective viscosity at reduced pressure (\( p \ll 50 \text{ Torr} \)):

\[
\mu = (\mu_p) py_o = (3.7 \times 10^{-2}) py_o
\]

units of \( \mu_p \) are \( \text{kg-s}^{-1}\text{-m}^{-2}\text{-Torr}^{-1} \)

units fixed for \( p \) in Torr

William Clark, Ph.D., EECS Dept., UC Berkeley, 1997

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**Squeeze-Film Damping**

- Plates slide in \( y \) direction

\[
b \approx 7\mu A z_o^2 / y_o^3
\]

Note that it’s just \( \mu \), not \( \mu_p \)
Mass-Spring-Damper System

\[ F(t) = F \cos(\omega t) \]
\[ x(t) = X \cos(\omega t) \]

\[ M \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = F(t) \]

Sinusoidal, steady-state response: \( X(t) = X e^{j\omega t} \)

\[ -M \omega^2 X e^{j\omega t} + j\omega b X e^{j\omega t} + k X e^{j\omega t} = F e^{j\omega t} \]

Second-Order Resonant Transfer Function

Solve for the ratio of the phasor displacement \( X \) to the phasor driving force \( F \)

\[ H(j\omega) = \frac{X}{F} = \frac{k^{-1}}{1 - \left(\frac{\omega}{\omega_1}\right)^2 + j \frac{\omega}{Q \omega_1}} \]

\[ \omega_1 = \sqrt{\frac{k}{M}} \quad Q = \frac{M \omega_1}{b} \]

Above analysis is for the light damping case (\( Q >> 1 \))
Limiting Cases

Low frequency: \( \omega << \omega_1 \)

High frequency: \( \omega >> \omega_1 \)

Resonant frequency: \( \omega = \omega_1 \)

Peak width (-3 dB): \( \Delta \omega = \omega_1 / Q \)

Magnitude Bode Plot

\[ k^{-1} = 1 \text{ m/N} \]
\[ Q = 1000 \]
Phase Bode Plot

Phase\(H\)

\[\omega_0 - \Delta \omega \quad \omega_0 + \Delta \omega\]

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