Electromechanical Models for MEMS Resonators

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Outline

• Basic electromechanics for MEMS
  linear suspension analysis
  electrostatic actuators

• Circuit models for micromechanical resonators
  motional elements
  parasitic elements

• Case study 1: comb-drive resonator

• Case study 2: radial bulk annual resonator
The Cantilever with a Concentrated Load

Clamped: \( y = 0, \frac{dy}{dx} = 0 \) at \( x = 0 \)

Find the tip deflection \( y(x = L_c) \) and applied load \( F \) … find effective spring constant \( k_c \)

\[
-M_z(x) = EI_z \frac{d^2y}{dx^2}
\]

The moment varies linearly with \( x \)

\[
-M_z(x) = F(L_c - x)
\]
Tip Deflection

- Integrate ODE twice and apply boundary conditions (zero displacement, zero slope) at anchor

\[ y(x) = \left( \frac{F}{6EIZ} \right) (3L_c - x)x^2 \]

- Tip deflection: \( y(L_c) \)

\[ y(L_c) = \left( \frac{F}{3EIZ} \right) L_c^3 \]

- Spring constant: \( k_c \) (N/m) … = (\( \mu \)N/ \( \mu \)m)

\[ k_c = \left( \frac{3EIZ}{L_c^3} \right) = \frac{EWh^3}{4L_c^3} \]
Summary of Common Loading and Boundary Conditions

Compendium of useful results: http://www.roarksformulas.com

<table>
<thead>
<tr>
<th></th>
<th>cantilever</th>
<th>guided-end</th>
<th>fixed-fixed</th>
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<tbody>
<tr>
<td>$x$</td>
<td>$\frac{F_\sigma L}{E h w}$</td>
<td>$\frac{F_\sigma L}{E h w}$</td>
<td>$\frac{F_\sigma L}{4E h w}$</td>
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<tr>
<td>$y$</td>
<td>$4 \frac{F_\sigma L^3}{E h w^3}$</td>
<td>$\frac{F_\sigma L^3}{E h w^3}$</td>
<td>$\frac{1}{16} \frac{F_\sigma L^3}{E h w^3}$</td>
</tr>
<tr>
<td>$z$</td>
<td>$4 \frac{F_\sigma L^3}{E w h^3}$</td>
<td>$\frac{F_\sigma L^3}{E w h^3}$</td>
<td>$\frac{1}{16} \frac{F_\sigma L^3}{E w h^3}$</td>
</tr>
</tbody>
</table>


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Series Combinations of Cantilevers

- Springs in series $\rightarrow$ same load; deflections add

$$L = 2L_c$$

$$y(L) = F/k = 2$$

$$y(L_c) = 2 \left( F/k_c \right) = F \left( 1/k_c + 1/k_c \right)$$

compliances add

$$1/k = 1/k_c + 1/k_c$$
Parallel Combinations of Springs

• Same displacement $\rightarrow$ load is shared and the spring constant is the *sum* of the individual spring constants.

$$y(L) = \frac{F}{k} = \frac{F_a}{k_a} = \frac{F_b}{k_b} = \frac{F}{2} \left(\frac{1}{k_a}\right)$$

$$k = 2k_a$$
The Lateral Resonator as a “Two-Port”

Overall Spring Constant

- Four pairs of clamped-guided beams, each of which bend in series (assume that trusses are inflexible)

Force is shared by each pair \( F_{\text{pair}} = F/4 \)

Displacement of two legs add (springs in series)

\[ y = F_{\text{pair}}/k_{\text{pair}} = (F/4)(1/k_{\text{leg}} + 1/k_{\text{leg}}) \]

\[ 1/k_{\text{leg}} = 1/k_c + 1/k_c = 2/k_c \]

\[ y = (F/4)(2/k_c + 2/k_c) = F/k_c \]
Two ways to change the energy:

1. Change the charge $q$
2. Change the separation $g$

$$\Delta W(q,g) = V \Delta q + F_e \Delta g$$

Note: we assume that the plates are supported elastically, so they don’t collapse.

Voltage-Control Case

- Practical situation: we control $V$ (since charge-control on the typical sub-pF MEMS actuation capacitor is problematic)
- Can we find $F_e$ as a partial derivative of the stored energy with respect to $g$ with $V$ held constant? **No!**
- Answer: apply Legendre transformation and define the co-energy $W'(V,g)$

$$W'(V,g) = qV - W$$

$$dW'(V,g) = qdV + Vdq - dW$$

$$dW'(V,g) = qdV + Vdq - (Vdq + F_e dg)$$

$$dW'(V,g) = qdV - F_e dg$$
Voltage-Control Case (Cont.)

- Evaluate the co-energy:

\[ W' = 0 + \int_0^V q(g,V')dV' \]

- Voltage in terms of charge at fixed gap:

\[ V' = \frac{\int_0^g Edg'}{\varepsilon A} = \left( \frac{q}{\varepsilon A} \right) g \]

\[ q(g,V') = \frac{\varepsilon A}{g} V' = C(g,A)V' \]

zero voltage \( \rightarrow \) zero force, zero integral

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Electrostatic Force (Voltage Control)

- Find co-energy in terms of voltage

\[ W' = \int_{0}^{V} q(g, V')dV' = \int_{0}^{V} \left( \varepsilon \frac{A}{g} \right) V'dV' = \frac{1}{2} \left( \frac{\varepsilon A}{g} \right) V^2 \]

- Variation of co-energy with respect to gap yields e.s. force:

\[ F_e = - \frac{\partial W'(V, g)}{\partial g} \bigg|_V = - \frac{1}{2} \left( \frac{\varepsilon A}{g^2} \right) V^2 = \frac{1}{2} \frac{C}{g} V^2 \]

\[ \text{strong function of gap!} \]

- Variation of co-energy with respect to voltage yields charge:

\[ q = \frac{\partial W'(V, g)}{\partial V} \bigg|_g = \left( \frac{\varepsilon A}{g} \right) V = CV \quad \text{as expected} \]
Spring-Suspended Capacitor: Voltage-Control Case

\[ F_e = \frac{\partial W'(V, g)}{\partial g} \bigg|_{q} = \frac{1}{2} \frac{\varepsilon A}{g^2} V^2 \]

\[ F_e = k z \]

\[ g = g_o - z = g_o - \frac{F_e}{k} = g_o - \frac{\varepsilon A}{2kg^2} V^2 \]

\[ q = \frac{\partial W'(V, g)}{\partial V} \bigg|_{g} = CV = \frac{\varepsilon A}{g} V \]

Voltage increases \( \rightarrow \)
Gap decreases \( \rightarrow \)
Force increases \( \ldots \)

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Interdigitated Comb Drive

Common bias:
DC offset $V_P$ connected to shuttle through poly0 “ground plane”

William Tang, Ph.D. EECS Dept., 1990
(this device by Clark Nguyen, Ph.D. 1994)

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Electrostatic Force: a First Pass

W. C. Tang, Ph.D. EECS Dept., 1990

gap = g, thickness = t
L = finger length
x = overlap length

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First-Pass Electrostatic Force (Cont.)

- Neglect fringing fields
- Parallel-plate capacitance between stator and rotor

\[
W'(x, V_{rs}) = \int_{0}^{V_{rs}} q(x, V')dV' = \int_{0}^{V_{rs}} \left(2N \frac{\varepsilon_o x t}{g}\right)V'dV' = \frac{1}{2} \left(2N \frac{\varepsilon_o x t}{g}\right)V_{rs}^2
\]

\[
F_e = \frac{\partial W'}{\partial x} = \frac{1}{2} V_{rs}^2 \frac{dC_{rs}}{dx} = \frac{1}{2} V_{rs}^2 \left(2N \frac{\varepsilon_o t}{g}\right)
\]

independent of \(x\)!

- Have we forgotten anything? The substrate!
Relative Forces for Surface Microstructures

Comb drive (x-direction) 
\((V_1 = V_2 = V_s = 1V)\)

\[ F_{e,x} = \frac{\varepsilon_0 t}{g} V_s^2 \]

Differential || plate (y-direction) 
\((V_1 = 0V, V_2 = 1V)\)

\[ F_{e,y} = \frac{1}{2} \frac{\varepsilon_0 tx}{g^2} V_2^2 \]

\[ \frac{F_{e,y}}{F_{e,x}} = \frac{\varepsilon_0 tx}{\frac{2\varepsilon_0 t}{V_s^2}} \cdot \frac{1}{2 \cdot g} \]

|| plate wins big ... for surface MEMS

gap = \(g = 1 \, \mu m\),
thickness = \(t = 2 \, \mu m\)
finger length = \(L = 100 \, \mu m\)
overlap length \(x = 75 \, \mu m\)
Comb Drive Force:  a Second Pass

- Energy must include capacitance between the stator and rotor and the underlying ground plane, which is typically biased at the stator voltage $V_s$ (why?)
Comb-Drive Force with Ground Plane Correction

- Finger displacement changes capacitances from stator and rotor to the ground plane $\rightarrow$ modifies the electrostatic energy

$$F_{e,x} = \frac{\partial W'}{\partial x} = \frac{1}{2} \frac{dC_{sp}}{dx} V_s^2 + \frac{1}{2} \frac{dC_{rp}}{dx} V_r^2 + \frac{1}{2} \frac{dC_{rs}}{dx} (V_s - V_r)^2$$

Gary Fedder, Ph.D., pp. 119-122, 1994
Capacitance Expressions

• Consider case where $V_r = V_p = 0$ V
• $C_{sp}$ depends on whether or not fingers are engaged

$$C_{sp} = N[C'_{sp, e} x + C'_{sp, u} (L - x)]$$

$$C_{rs} = NC'_{rs} x$$

Gary Fedder, Ph.D., pp. 119-122, 1994
Simulation (2D Finite Element)

\[ F_{e,x} = \frac{N}{2} \left( C_{rs}' + C_{sp,e}' - C_{sp,u}' \right) V_s^2 \]

20-40% reduction of \( F_e \)

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Gary Fedder, Ph.D., p. 123, 1994
Vertical Force (Levitation)

Consider $V_r = 0$ V as shown:

$$F_{e,z} = \frac{1}{2} Nx \left[ \frac{d\left(C'_{sp,e} + C'_{rs}\right)}{dz} \right] V_s^2$$
Experimental Measurements

Shuttle is pulled down (toward the substrate) with zero applied voltage

Why?

Charged Dielectrics: No Applied Voltage Needed!

Nitride charge inferred from deflection and simulated field distribution is consistent with typical values

Comb-Drive Lateral Resonator

Anchor connects ground plane and resonator

Typical bias:
\[ V_I = V_O = 0 \, \text{V} \]

DC voltage across drive and sense electrodes to resonator = \( V_P \)

Input Current

Input current $i_1(t)$ is the derivative of the charge $q_1 = C_1 v_D$

$$i_1(t) = C_1 \frac{dv_D}{dt} + v_D \frac{dC_1}{dt}$$

$$v_D(t) = V_I + v_1(t) - V_p = -V_{p1} + v_1(t)$$

The capacitance $C_1$ has a DC component and a time-varying component due to the motion of the structure

$$C_1(t) = C_{o1} + C_{m1}(t) \quad C_{m1}(t) = \frac{\partial C_1}{\partial x} x(t) \quad \text{(linearized case)}$$

Substitute to find the input current:

$$i_1(t) = C_{o1} \frac{dv_1}{dt} + C_m \frac{dv_1}{dt} + (-V_{p1}) \frac{\partial C_1}{\partial x} \frac{\partial x}{\partial t} + v_1 \frac{\partial C_1}{\partial x} \frac{\partial x}{\partial t}$$

$$i_{1x}(t)$$
Input Motional Admittance $Y_{1x}(j\omega)$

Phasor form of the motional current $i_{1x}$:

$$I_{1x}(j\omega) = -V_{p1} \frac{\partial C_1}{\partial x} (j\omega X)$$

The input motional admittance (inverse of impedance) is the ratio of the phasor motional current to the ac drive voltage:

$$Y_{1x}(j\omega) = \frac{I_{x1}(j\omega)}{V_1(j\omega)} = -V_{p1} \frac{\partial C_1}{\partial x} j\omega \left( X(j\omega) \right) \left( \frac{1}{V_1(j\omega)} \right)$$

The displacement-to-voltage ratio can be re-expressed in terms of the drive force $F_d(j\omega)$

$$Y_{1x}(j\omega) = -V_{p1} \frac{\partial C_1}{\partial x} j\omega \left( X(j\omega) \right) \left( \frac{F_d(j\omega)}{V_1(j\omega)} \right)$$
Input Admittance (Cont.)

The electrostatic force component at the drive frequency $\omega$ is:

$$f_{d,\omega}(t) = \frac{1}{2} v_D^2(t) \left. \frac{\partial C_1}{\partial x} \right|_{\omega} = -V_{p1} v_1(t) \frac{\partial C_1}{\partial x} \quad \rightarrow \quad \frac{F_d(j\omega)}{V_1(j\omega)} = -V_{p1} \frac{\partial C_1}{\partial x}$$

The 2nd order mechanical response of the resonator is

$$\frac{X(j\omega)}{F_d(j\omega)} = \frac{k^{-1}}{1 - (\omega / \omega_o)^2 + j(\omega / Q\omega_o)}$$

The input admittance is:

$$\frac{I_{lx}(j\omega)}{V_1(j\omega)} = \left( -V_{p1} \frac{\partial C_1}{\partial x} j\omega \right) \left[ \frac{k^{-1}}{1 - (\omega / \omega_o)^2 + j(\omega / Q\omega_o)} \right] \left( -V_{p1} \frac{\partial C_1}{\partial x} \right)$$

$$\frac{I_{lx}(j\omega)}{V_1(j\omega)} = \frac{j \omega k^{-1} V_{p1}^2 \left( \frac{\partial C_1}{\partial x} \right)^2}{1 - (\omega / \omega_o)^2 + j(\omega / Q\omega_o)}$$
Series $L$-$C$-$R$ Admittance

The current through an $L$-$C$-$R$ branch is:

\[
I(j\omega) = \frac{j\omega C}{V(j\omega)} = \frac{j\omega C}{\frac{1}{\omega^2} + j(\omega RC)}
\]

\[
\omega_o^{-2} = LC
\]

Match terms in motional admittance → find equivalent elements
Equivalent Circuit for Input Port

A series L-C-R circuit results in the identical expression → find equivalent values \( L_{x1}, C_{x1}, \) and \( R_{x1} \)

\[
L_{x1} = \frac{m}{\eta^2} \quad C_{x1} = \frac{\eta^2}{k} \quad R_{x1} = \frac{\sqrt{k m}}{Q \eta^2}
\]

\[\eta = V_{p1} \frac{\partial C_1}{\partial x} = \text{electromechanical coupling coefficient}\]

At resonance, the impedances of the inductance and the capacitance cancel out →

\[
I_{x1} = \frac{V_1}{R_{x1}}
\]
Output Port Model

Consider first the current due to driving the input (set $v_2 = 0$ V)

$$i_2(t) = -V_p2 \frac{\partial C_2}{\partial t} = -V_p2 \frac{\partial C_2}{\partial x} \frac{\partial x}{\partial t}$$

In phasor form,

$$I_2(j\omega) = j\omega V_p2 \frac{\partial C_2}{\partial x} X(j\omega) = \frac{j \omega k^{-1} V_p1 V_p2}{1 - (\omega / \omega_o)^2 + j(\omega / Q \omega_o)} \left( \frac{\partial C_1}{\partial x} \right) \left( \frac{\partial C_2}{\partial x} \right) V_1(j\omega)$$

$I_2$ and $I_{x1}$ are related by the forward current gain $\phi_{21}$:

$$\phi_{21} = \frac{I_2(j\omega)}{I_{x1}(j\omega)} = \frac{V_p2}{V_p1} \frac{\partial C_2}{\partial x} \frac{\partial C_1}{\partial x} \rightarrow \text{model by a current-controlled current source}$$
Two-Port Equivalent Circuit ($v_2 = 0$)
Complete Two-Port Model

Symmetry implies that modeling can be done from port 2, with port 1 shorted → superimpose the two models.
Equivalent Circuit for Symmetrical Resonator ($\phi_{21} = \phi_{12} = 1$)

C. T.-C. Nguyen, Ph.D., UC Berkeley, 1994

$C_x = 0.5 \text{ fF}$
$L_x = 200 \text{ kH}$
$R_x = 500 \text{ k}\Omega$
$C_{oi}, C_{oo} = 15 \text{ fF}$
455 kHz Comb-Drive Resonator Values

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
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<td>$E$</td>
<td>150</td>
<td>GPa</td>
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<tr>
<td>$W$</td>
<td>2</td>
<td>μm</td>
</tr>
<tr>
<td>$L$</td>
<td>26.4</td>
<td>μm</td>
</tr>
<tr>
<td>$h$</td>
<td>2</td>
<td>μm</td>
</tr>
<tr>
<td>$d$</td>
<td>1</td>
<td>μm</td>
</tr>
<tr>
<td>$N$</td>
<td>40 (20 shuttle fingers)</td>
<td>—</td>
</tr>
<tr>
<td>$M_p$</td>
<td>$3.1 \times 10^{-11}$</td>
<td>kg</td>
</tr>
<tr>
<td>$f_o$</td>
<td>455</td>
<td>kHz</td>
</tr>
<tr>
<td>$Q$</td>
<td>50 000</td>
<td>—</td>
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<tr>
<td>$V_p$</td>
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<td>V</td>
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<td>H</td>
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<td>aF</td>
</tr>
<tr>
<td>$C_o$</td>
<td>7.1</td>
<td>fF</td>
</tr>
</tbody>
</table>

← assumes vacuum
← not small
← huge!
← mind-boggling!

C. T.-C. Nguyen, Ph.D., UC Berkeley, 1994
Double-Ended Tuning Fork Resonators

Current through structure $\rightarrow$ more resistance (decreases Q)
more feedthrough to substrate

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T. A. Roessig, Ph.D., ME, UC Berkeley, 1997
Ideal Tuning Fork Two-Port Response

Phase change of 180° at resonance “pins” the frequency, with drifts in the feedback amplifier having little effect

Response assumes no feedthrough capacitance between input and output ports

T. Roessig, Ph.D., ME, UC Berkeley, 1997
Tuning Fork Response with Capacitive Feedthrough $C_f$

Feedthrough capacitance results in a null in the amplitude response and an added sense current which increases with frequency … and which can obscure the resonance entirely!

Next lecture: $C_f$ and its control

T. A. Roessig, Ph.D., ME, UC Berkeley, 1997
Radial Bulk Annular Resonator (RBAR)


\[ r_{av} = \frac{r_o + r_i}{2} \]

\[ W_r = r_o - r_i \]
1.14 GHz Poly-Si Disk Resonator

- Note $Q$ in vacuum and in air is the same: little energy loss to ambient
- Energy loss through anchor ("stem") is significant

Fig. 7: Measured (dark) and predicted (light) frequency characteristics for a 1.14-GHz, 3rd mode, 20μm-diameter disk resonator measured in (a) vacuum and (b) in air, using a mixing measurement setup.
Two-Port Resonator Model

$R_x$ is relatively large due to inefficient transduction
$C_o$ can be relatively large
$C_f$ (feedthrough capacitance) shunts the resonator
RBAR Equivalent Circuit

- $Q = 10,000$
- Gap = $g = 50$ nm
- Film thickness $t = 2$ $\mu$m
- Radius = $r_{av} = 50$ $\mu$m
- Static capacitance = $C_o = 10$ fF
- DC Bias = 10 V

- Neglect interconnect capacitance and parasitic capacitance to ground plane

- Find impedance of the device as a one-port (ground sense terminal)
100 MHz RBAR Impedance

Peter Chen, BSAC

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500 MHz RBAR Impedance

Peter Chen, BSAC
Capacitive Interface Design

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Outline

• The capacitor as a position sensor
  parasitic electrostatic forces
  differential excitation
  capacitive half-bridge
  position resolution

• Buffer amplifier topologies for capacitive sensors
  basic implementation using negative feedback
  bootstrapping the input capacitance
  transresistance amplifiers

• Electronic and mechanical noise sources
The Simple Capacitor Divider

\[ v(t) = \hat{V} \cos(\omega t) \]

Why modulate \( v(t) \)?

\[ V_{out} = \hat{V} \left( \frac{Z_{ref}}{Z_{ref} + Z(x)} \right) \]

Ideal buffer: \( C_{in} = 0 \)

\[ V_{out} = \hat{V} \left( \frac{1}{\frac{1}{j\omega C_{ref}} + \frac{1}{j\omega C(x)}} \right) = \hat{V} \left( \frac{1}{1 + \frac{C_{ref}}{C_{ox}}} \right) \]
A Capacitive Divider from 1980

Metal gate of MOSFET is directly connected to the top plate of the sense capacitor $V_B$ .. other capacitor $C_p$ is parasitic

**Question:** how is the potential $V_B$ set?

**Answer:** with an external probe tip!

…the two-transistor amplifier would remain biased for high gain for a few minutes after $V_B$ was set

Matched Air-Gap Reference Capacitors

compliant suspension (vertical $a_z$ sensitivity)

$C(x) = \frac{\varepsilon_o A}{g_o + x}$

stiff suspension: insensitive

$C_{\text{ref}} = \frac{\varepsilon_o A}{g_o}$

Weijie Yun, P. R. Gray, and R. T. Howe, Hilton Head Workshop, 1992, pp. 21-25.

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Simple Capacitor Divider (Cont.)

Offset signal is undesirable for buffer amplifier and for downstream signal processing.

\[ V_{out} = \hat{V} \left( \frac{1}{1 + \frac{\varepsilon_o A}{\varepsilon_o A/(g_o + x)}} \right) = \hat{V} \left( \frac{1}{1 + \frac{(g_o + x)}{g_o}} \right) = \hat{V} \left( \frac{g_o}{2g_o + x} \right) \]

\[ V_{out} = \hat{V} \left( \frac{g_o}{2g_o} \right) \left( \frac{1}{1 + x/2g_o} \right) \approx \frac{\hat{V}}{2} - \frac{\hat{V}}{2} \left( \frac{x}{2g_o} \right) \]

matched air-gap reference capacitor

offset

signal
Capacitor Divider With Differential Excitation

Why modulate $v_+$ and $v_-$?

Ideal buffer: $C_{in} = 0$

Impedance divider with superposition:

$$V_{out} = \hat{V} \left( \frac{Z_{ref}}{Z_{ref} + Z(x)} \right) - \hat{V} \left( \frac{Z(x)}{Z_{ref} + Z(x)} \right)$$
Improved Capacitive Divider (Cont.)

\[ V_{out} = \hat{V} \left( \frac{Z_{ref} - Z(x)}{Z_{ref} + Z(x)} \right) = \hat{V} \left( \frac{C_{ref}^{-1} - C^{-1}(x)}{C_{ref}^{-1} + C^{-1}(x)} \right) = \hat{V} \left( \frac{g_o - (g_o + x)}{g_o + x + g_o} \right) \]

\[ V_{out} = -\hat{V} \left( \frac{x}{2g_o + x} \right) \approx -\hat{V} \left( \frac{x}{2g_o} \right) \] no offset!

\[ \nu_{out}(t) = -\hat{V} \left( \frac{x(t)}{g_o + x(t)} \right) \cos(\omega t) = -\hat{V} \left( \frac{x(t)}{2g_o} - \left( \frac{x(t)}{2g_o} \right)^2 + \ldots \right) \cos(\omega t) \] distortion
Parasitic Electrostatic Force for Differential Excitation

\[ f(t) = \frac{1}{2} (v_+ - v_{out})^2 \left( \frac{dC}{dx} \right) \approx \frac{1}{2} \hat{V}^2 \cos^2(\omega t) \left( \frac{-\varepsilon_0 A}{g_o^2} \right) \]

For small displacements, \( v_{out} \approx 0 \) V.

Force has both DC and \( 2\omega \) components: pull-in and resonant excitation can happen!
The Capacitive Half-Bridge

\[ v_+(t) = \hat{V} \cos(\omega t) \]

\[ C_+(x) = \varepsilon_o A / (g_o + x) \]

\[ C_-(x) = \varepsilon_o A / (g_o - x) \]

\[ v_-(t) = -\hat{V} \cos(\omega t) \]

Impedance divider with superposition:

\[ V_{out} = \hat{V} \left( \frac{Z_+(x)}{Z_+ + Z_-(x)} \right) - \hat{V} \left( \frac{Z_-(x)}{Z_- + Z_+(x)} \right) \]
Capacitance Half Bridge (Cont.)

Simplify expression:

\[ V_{out} = \hat{V} \left( \frac{g_o + x}{2g_o} \right) - \hat{V} \left( \frac{g_o - x}{2g_o} \right) = \frac{\hat{V}}{2} \frac{2x}{g_o} = \hat{V} \left( \frac{x}{g_o} \right) \]

no offset; 2X signal increase

Electrostatic force:

\[ f(t) = \frac{1}{2} \left( v_+ - v_{out} \right)^2 \left( \frac{dC_+}{dx} \right) - \frac{1}{2} \left( v_{out} - v_- \right)^2 \left( \frac{dC_-}{dx} \right) \]

\[ f(t) = \frac{1}{2} \left( v_+ - v_{out} \right)^2 \left( \frac{-C_o}{g_o} \right) - \frac{1}{2} \left( v_{out} - v_- \right)^2 \left( \frac{+C_o}{g_o} \right) \]
Electrostatic Force (Cont.)

\[ f(t) = \frac{1}{2} \left( \frac{C_o}{g_o} \right) \left[ \left( \hat{V} \cos \omega t - v_{out} \right)^2 - \left( v_{out} - \hat{V} \cos \omega t \right)^2 \right] \]

\[ f(t) = \frac{1}{2} \left( \frac{C_o}{g_o} \right) \left[ \left( \hat{V}^2 \cos^2 \omega t - 2v_{out} \hat{V} \cos \omega t + v_{out}^2 \right) - \left( v_{out}^2 - 2v_{out} \hat{V} \cos \omega t + \hat{V}^2 \cos^2 \omega t \right) \right] \]

\[ f(t) = 2 \left( \frac{C_o}{g_o} \right) v_{out} \hat{V} \cos \omega t \]

Output voltage is proportional to the displacement (for \( x << g_o \))

\[ v_{out} = \left( \frac{x}{g_o} \right) \hat{V} \cos \omega t \]

\[ f(t) = 2 \left( \frac{C_o}{g_o} \right) \left( \frac{x}{g_o} \right) \hat{V}^2 \cos^2 \omega t = \left( \frac{2C_o}{g_o^2} \right) \hat{V}^2 \cos^2 \omega t \]

DC and 2\( \omega \) terms
Electrostatic Spring Constant $k_e$

\[ v_+(t) = \hat{V} \cos(\omega t) \]

\[ f(t) = \left( \frac{2C_o}{g_o} \hat{V}^2 \cos^2 \omega t \right) x = -k_e x \]

\[ k_e = - \frac{2C_o}{g_o} \hat{V}^2 \cos^2 \omega t \]

note direction: spring applies force *opposite* to displacement

both DC and $2\omega$ components: use square wave excitation to yield constant $k_e$
Parasitic Capacitances
Surface micromachined z-axis parallel-plate capacitor
Equivalent Circuit

$C_{pp}(x)$: nominal $||$ plate sense capacitor
$C_{f1}(x)$: fringe capacitance (varies with plate displacement)
$C_{f2}$: fringe capacitance between upper plate (connected to anchor plane) and lower plate ... slight dependence on $x$
$C_{pu}$: parasitic capacitance from upper plate to substrate
$C_{pl}$: parasitic capacitance from lower plate to substrate

Velocity Sensing

Fundamental current-voltage relationship for a time-varying capacitor:

\[
i = \frac{dq}{dt} = \frac{d}{dt}[C_s(t)v_s(t)] = C_s(t)\frac{dv}{dt} + v_s(t)\frac{dC_s}{dt}
\]

Consider special case: \( v = V_P = \text{constant} \)

\[
i = V_P \frac{dC_s}{dt}
\]

… used in high-quality capacitance microphones
Velocity Sensing (Cont.)

Sense capacitor’s time variation:

\[
\frac{dC_s}{dt} = \frac{dC_s}{dx} \frac{dx}{dt} = \frac{dC_s}{dx} v
\]

Parallel-plate sense capacitor with gap \(g_o\):

\[
\frac{dC_s}{dx} \bigg|_{x=0} = -\frac{C_{so}}{g_o}
\]

Harmonic motion: \(x(t) = \hat{x} \cos \omega t\)

\[
i_s = V_p \frac{dC_s}{dt} = -V_p \frac{C_{so}}{g_o} (-\hat{x} \omega \sin \omega t) = \left(\frac{V_p C_{so} \hat{x} \omega}{g_o}\right) \sin \omega t
\]
Some Numbers

Surface micromachined capacitors:

- $C_s \approx 100 \, \text{fF}$
- $g_o = 1 \, \mu\text{m}$
- $V_+ = -V_- = 2.5 \, \text{V}$

\[ v_{out} = S_x x \]

\[ S_x = \frac{V_+}{g_o} = 2.5V / \mu\text{m} \]

\[ v_{out}\bigg|_{\text{min}} = 100 \mu\text{V} \quad \text{…noise in buffer amp} \]

\[ x_{\text{min}} = \frac{v_{out}\bigg|_{\text{min}}}{S_x} = \frac{100 \mu\text{V}}{2.5V / \mu\text{m}} = 40 \times 10^{-6} \mu\text{m} \]

\[ x_{\text{min}} = 40 \, \mu\text{m} \quad \text{is this real?} \]

ADXL-50 sense capacitor
World Record Capacitive Position-Sense Resolution*

Analog Devices ADRS-150 vibratory rate gyroscope
John Geen, Steve Sherman, John Chang, and Steve Lewis,

Full scale Coriolis-induced displacement = 20 Å

Sense capacitance \(\approx 1000 \text{ fF}\)

Minimum detectable capacitance change \(\approx 12 \text{ zF} = 0.012 \text{ aF}\)

Nominal sense gap = 1.6 \(\mu\text{m}\)

Minimum displacement: 16 fm!

*surface micromachining class audio frequency band
Averaging \( \rightarrow \) Extreme Resolution

At \( V_+ = 5 \text{ V} \), the charge on the sense capacitor is:

\[
q_s = C_s V_+ = (1000 \text{ fF})(5 \text{ V}) = 5000 \text{ fC}
\]

Number of electrons at \( 1.6 \times 10^{-19} \text{ C/electron} \):

\[
N_s = 3.125 \times 10^7
\]

Minimum detectable change in sense charge:

\[
\Delta q_s = \Delta C_{s, \text{min}} V_+ = (12 \text{ zF})(5 \text{ V}) = 60 \text{ zC}
\]

Minimum detected change in number of electrons:

\[
\Delta n_s = \Delta q_s / q_e = 60 \times 10^{-21} / 1.6 \times 10^{-19} \approx 0.4
\]
The Capacitive Half-Bridge, Revisited

How do we design a 1X buffer with $C_{in} \approx 0$?

$$v_+(t) = \hat{V} \cos(\omega t)$$

$$C_+(x) = \varepsilon_o A / (g_o + x)$$

$$v_-(t) = -\hat{V} \cos(\omega t)$$

$$C_-(x) = \varepsilon_o A / (g_o - x)$$

$$V_{out} = \hat{V} \left( \frac{Z_+(x)}{Z_+ + Z_- (x)} \right) - \hat{V} \left( \frac{Z_-(x)}{Z_- + Z_+ (x)} \right)$$

$$V_{out} = V_{in} = \hat{V} \left( \frac{x}{g_o} \right)$$
Precision Unity Gain via Feedback

\[ V_{out} = A_d (v_+ - v_-), \quad A_d \text{ is large} \]

\[ V_{+} = V_{in} \]

\[ V_{-} = V_{out} \]

\[ V_{out} = A_d (v_{in} - v_{out}) \]

\[ V_{out} = \left( \frac{A_d}{1 + A_d} \right) v_{in} \approx v_{in} \]

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Basic CMOS Differential Amplifier

Simplified analysis:

\[ v_+ = - v_- = \frac{(v_+ - v_-)}{2} \]

Nodes \(a\) and \(b\) are incremental grounds.
Differential Gain $A_d$

\[ v_{\text{out}} = (-g_{m2}v_{gs2})r_{o2} \]
\[ v_{\text{out}} = -g_{m2}r_{o2}v_- \]
\[ v_{\text{in.d}} = v_+ - v_- = -2v_- \]
\[ v_{\text{out}} = -g_{m2}r_{o2}(-v_{\text{in.d}}/2) \]
\[ v_{\text{out}} = (g_{m2}r_{o2}/2)v_{\text{in.d}} \]

Typical values

\[ g_{m2}r_{o2} = 200 \]
\[ A_d = 100 \]
Basic Unity-Gain Buffer

![Diagram of a basic unity-gain buffer circuit]

- **Input resistance** $R_{in} = \infty \Omega$
- **Input capacitance** $C_{in} = ?$

- Short-circuit output to inverting input

USA-Korea MEMS Symposium July 2004
Input Capacitance $C_{in}$

Resistor $R' = (g_{m2})^{-1} \ || \ r_{o2} + r_{o,cs} \to \infty$

Therefore, $V_{gs1} \approx 0 \ V \rightarrow$ phasor current through $C_{gs1} \approx 0 \ A \ldots C_{gs1}$ doesn’t exist (it’s been “bootstrapped”). In practice, $C_{gs1}$ is reduced 100 fold.

Input capacitance is due to gate-drain capacitance of $M_1$: $C_{in} \approx C_{gd1}$

USA-Korea MEMS Symposium July 2004
Improved Unity-Gain Buffer

Weijie Yun, Ph.D. EECS, 1992

$M_3, M_4$ cause $V_{d1}$ to track $V_{in}$ → $C_{gd1}$ is bootstrapped, too!

Result: $C_{in}$ approaches zero.
Setting the DC Bias

\[ v_+ (t) = \hat{V} \cos(\omega t) \]

\[ v_- (t) = -\hat{V} \cos(\omega t) \]

Node A has no path to ground; so it’s called a “floating node”

**Solutions:**

1. W. Yun and G. K. Fedder: “zero-biased diode” to leak \( A \) to a known DC voltage (e.g., \( V_{DD}/2 \))
2. W. Clark, M. Palaniapan: MOSFET biased in sub-threshold to set \( A \) to DC potential of \( M_1 \)’s drain.
3. M. Lemkin: MOSFET switch

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Current-to-Voltage (or Transresistance) Amplifier

\[ v_{out} \approx -R_f i_{in} \]

The feedback resistor can be implemented using a MOSFET biased in the triode region.
**Microresonator Oscillator Schematic**

**Transresistance Amplifier:**
- $M_3$ implements a variable resistance $R_f$
- $M_1$-$M_2$ implement a simple inverting amplifier
- $M_6$-$M_7$ implement a second amplifying stage

Integrated 16.5 kHz Microresonator Oscillator


Erratic (chaotic) behavior observed for high DC biases in this and other MEMS oscillators was later explained by Kim Turner (Ph.D. Cornell, 1999, now UCSB).

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Electronic Noise Sources

1. Thermal noise in resistors
generated by random motion of electrons or holes
white spectral density (up to 10 THz)

\[ \langle v_n^2 \rangle = 4k_BTR\Delta f \]

spectral density

\[ \langle v_n^2 \rangle / \Delta f = 4k_BTR \]

Example: \( R = 1k\Omega, T = 300 \text{ K} \)

\[ \sqrt{\langle v_n^2 \rangle / \Delta f} \approx 4nV / \sqrt{\text{Hz}} \]

\[ v_{rms} = \sqrt{\langle v_n^2 \rangle} \approx 4mV \]

(in a 1 MHz bandwidth)
Thermal Noise (Cont.)

Thermal noise current: find Norton equivalent

\[ \langle i_n^2 \rangle / \Delta f = \frac{4k_B T}{R} \]

Example: \( R = 1 \, \text{k}\Omega, \, T = 300 \, \text{K}, \, BW = 1 \, \text{MHz} \)

\[ \sqrt{\langle i_n^2 \rangle} = \sqrt{\frac{4k_B T}{R}} \sqrt{BW} = 4 \, \mu\text{A} \]
Resistor Noise in MEMS

Interconnect resistance $R_{int}$ to capacitive position sensors

> routing in polySi$_0$ can lead to a high resistances, due to relatively high sheet resistance of this layer

> inertial MEMS often have compliant suspensions $\rightarrow$ large number of squares in polySi$_1$ and significant contribution to $R_{int}$

$$v_+ (t) = \hat{V} \cos(\omega t)$$

$$v_- (t) = -\hat{V} \cos(\omega t)$$

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Flicker \((1/f)\) Noise

Noise mechanism requires a DC current, in contrast to thermal noise

Origin of \(1/f\) noise in MOSFETs: surface states

\[
\langle i_n^2 \rangle = \frac{KI^a}{f} \Delta f
\]

Near DC, the noise current diverges!
MOSFET Noise Sources

When biased in saturation, \( (V_{DS} > V_{DS,\text{sat}}) \), the noise can be represented by an input noise voltage and an input noise current.

Origin of 1/f noise in MOSFETs: surface states

\[
\langle v_{in}^2 \rangle = 4k_B T \left( \frac{2}{3} \frac{1}{g_m} \right) \Delta f + \left( \frac{K_f}{WLC_{ox}} \right) \frac{\Delta f}{f}
\]

channel resistance in saturation

\[
\langle i_{in}^2 \rangle \approx \left( \frac{\omega^2 C_{gs}^2}{g_m^2} \right) \left[ 4k_B T \frac{2}{3} \frac{1}{g_m} \Delta f + \frac{Kl_D}{f} \Delta f \right]
\]

inverse dependence on gate area \( \rightarrow \)
larger transistors have lower 1/f noise

(neglecting DC gate current and it’s shot noise)
MOSFET Noise Sources (Cont.)

Equivalent MOSFET small-signal model with input-referred noise sources

\[ \bar{v}_{in}^2 \quad \bar{i}_{in}^2 \]

Crossover frequency between thermal and flicker noise can range from 1 kHz to 1 MHz
Buffer Equivalent Input Noise

Substitute noise voltage and current for each MOSFET in buffer and find the total equivalent noise at the input

\[
\langle v_{in}^2 \rangle = 4k_B T \left( \frac{4}{3} \frac{1}{g_{m1}} \right) \left( 1 + \sqrt{\frac{g_{m6}}{g_{m1}}} \right) \Delta f + \ldots \text{ flicker noise terms}
\]

\[
v_+ (t) = \hat{V} \cos(\omega t)
\]

\[
v_- (t) = -\hat{V} \cos(\omega t)
\]
Mechanical (Brownian) Noise

Impinging molecules give rise to a Brownian noise force:

\[ \left\langle f_n^2 \right\rangle = 4k_B Tb \Delta f \]

\[ b = (M\omega_1)/Q = \text{damping coefficient} \]

Noise force applied to \( M-k-b \) system results in random Brownian motion with a \textit{frequency-dependent} power spectrum:

\[ \left\langle x_n^2 \right\rangle = \frac{(4k_B Tb / k)\Delta f}{\left[1 - (f / f_1)^2\right]^2 + \left(\frac{f}{Qf_1}\right)^2} \]

Implications: \( f << f_1 \rightarrow \)
Combining Noise Sources

If noise sources are uncorrelated, then their powers can be added. For two voltage noise sources:

\[
\langle v_n^2 \rangle = \langle v_{n1}^2 \rangle + \langle v_{n2}^2 \rangle
\]

\[
\sqrt{\langle v_n^2 \rangle} = v_{rms} = \sqrt{\langle v_{n1}^2 \rangle + \langle v_{n2}^2 \rangle}
\]

For a sensor, it is convenient to refer all noise sources to the input, by scaling them appropriately. For a capacitive divider position sensor, the position noise due to electronic noise at the input of the buffer is:

\[
\langle v_n^2 \rangle = \hat{V}^2 \left( \frac{\langle x_{n,e}^2 \rangle}{g_o^2} \right) \quad \rightarrow \quad \langle x_{n,e}^2 \rangle = g_o^2 \left( \frac{\langle v_n^2 \rangle}{\hat{V}^2} \right)
\]
SNR and DR, Defined

Signal-to-noise ratio = \( SNR \)

\[
SNR = 10 \log \left( \frac{P_s}{P_n} \right) = 10 \log \left( \frac{\langle v_s^2 \rangle}{\langle v_n^2 \rangle} \right) = 20 \log \left( \frac{v_s}{\sqrt{\langle v_n^2 \rangle}} \right) = 20 \log \left( \frac{v_s}{v_{n,rms}} \right)
\]

Note 1: \( P_{\text{noise}} \) is calculated over a limited bandwidth

Note 2: \( v_{n,rms} \) is taken as the minimum detectable signal, in the absence of special coding or signal processing

Dynamic range = \( DR \)

\[
DR = 10 \log \left( \frac{P_{s,\text{max}}}{P_{\text{noise}}} \right) = 10 \log \left( \frac{\langle v_{s,\text{max}}^2 \rangle}{\langle v_n^2 \rangle} \right) = 20 \log \left( \frac{v_{s,\text{max}}}{v_{n,rms}} \right)
\]
Signal and Noise Waveforms

Sinusoid with amplitude normalized to 1

Gaussian noise with rms level normalized to 1; note that peak-peak noise level is occasionally as high as 6!
SNR = 1 and 10 for Gaussian Noise